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IMPROVED QUANTUM RECEIVERS FOR OPTICAL COMMUNICATION.(U)

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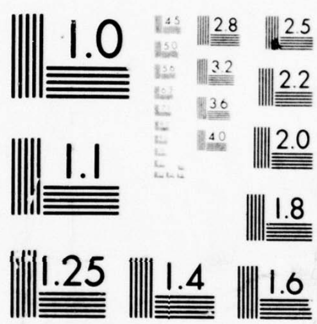

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Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Final Report

6 **Improved Quantum Receivers for Optical Communication.**

Principal Investigator – Robert S./Kennedy
10 Horace P./Yuen

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Contract N00014-76-C-0605

9 Final rept.

Covering the period 1 Jan~~uary 1,~~ 1976 – Jun~~e 30,~~ 1977

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FINAL REPORT

Improved Quantum Receivers for Optical Communication

Contract N00014-76-C-0605

Covering the period January 1, 1976 - June 30, 1977

The following is a brief summary of the work performed in this contract and also indicates certain important open problems. More details can be found in the attached papers.

I. SUMMARY OF RESEARCH

Our work was motivated by the potential of greatly improved optical communication that would result from the use of a new class of radiation quantum states. These states have been named two-photon coherent states (TCS)[(1)], and a thorough study of their properties has been completed[(1)-(2)]. In particular, their possible generation by degenerate parametric processes and two-photon lasers has been investigated. Realization of the quantum measurement that leads to the predicted quantum noise reduction has also been found [(3)-(4)]. The point process counting statistics of TCS's and their performance in conventional optical reception were worked out[(5)].

The main topic of this contract is on the communication theoretic aspects of TCS (Two-photon coherent states) systems. ~~We have shown~~ The optimality of TCS has been shown as transmitter quantum states, in a number of linear modulation and binary detection problems. ~~(6)-(8), (5)~~

Grant Sayer

~~We have also developed~~ a theory of quantum field propagation ~~(8)~~ has also been developed which is essential for a correct analysis of TCS behavior in optical transmission systems. In particular, ~~we now understand~~ the difference between near-field and far-field TCS communication is now understood. While one can realize the full TCS performance potential in near-field communication, the far-field case is uncertain at present.

II. SUGGESTIONS

The most important open problems relating to TCS communication can be identified, at this stage, to be the following:

1. Investigation into the possiblity of employing TCS-state-generating local oscillators for greatly improved far-field optical communication.
2. To design and subsequently perform an experiment which demonstrates that the desired quantum states, TCS's, can be produced.
3. To ascertain the engineering requirements and sensitivity tolerance in a practical TCS communication system, for both near and far field transmission.
4. To develop the general quantum communication theory of optimum state selection in different communication situations.

III. PUBLICATIONS UNDER THIS CONTRACT

1. H.P. Yuen, "States that give the maximum signal-to-noise ration for a fixed energy", Physics Letters, vol. 56A, pp. 105-106, 1976.

2. H.P. Yuen, "Two-photon coherent states of the radiation field", Physical Review A, vol. 13, pp. 2226-2243, 1976.

3. H.P. Yuen, "Gaussian states and parametric processes," to be submitted to Physical Review.

4. H.P. Yuen and J.H. Shapiro, "Optical communication with two-photon coherent states - Part I: quantum state propagation and quantum noise reduction", to be published in IEEE Transactions on Information Theory.

5. H.P. Yuen and J.H. Shapiro, "Optical communication with two-photon coherent states - Part III: quantum measurements realizable with photoemissive detectors", to be submitted to IEEE Transactions on Information Theory.

<u>REFERENCES</u>

(1) H.P. Yuen, "Two-photon coherent states of the radiation field", Physical Review A, vol. 13, pp. 2226-2243, 1976.

(2) H.P. Yuen, "Gaussian states and parametric processes", to be submitted to Physical Review.

(3) H.P. Yuen and J.H. Shapiro, "Quantum statistics of homodyne and heterodyne detection", to be published in <u>Coherence and Quantum Optics</u>, Proceedings of the Fourth Rochester Conference on Quantum Optics, Plenum, New York, 1978.

(4) H.P. Yuen and J.H. Shapiro, "Optical communication with two-photon coherent states - Part III: quantum measurements realizable with photoemissive detectors", to be submitted to IEEE Transactions on Information Theory.

(5) J.H. Shapiro, H.P. Yuen, and J.A. Machado Mata, "Optical communication with two-photon coherent states - part II: photoemissive detection and structured receiver performance", to be published in IEEE Transactions on Information Theory.

(6) H.P. Yuen, "States that give the maximum signal-to-noise ratio for a fixed energy", Physics letters, vol. 56A, pp. 105-106, 1976.

(7) H.P. Yuen, "Two-photon coherent states and state optimization in quantum communication theory", in Abstracts of Papers, Part I, Fourth International Symposium on Information Theory, Leningrad, pp. 228-230, 1976.

(8) H.P. Yuen and J.H. Shapiro, "Optical communication with two-photon coherent states - Part I: quantum state propagation and quantum noise reduction", to be published in IEEE Transactions on Information Theory.

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STATES THAT GIVE THE MAXIMUM SIGNAL-TO-QUANTUM NOISE RATIO FOR A FIXED ENERGY☆

H.P. YUEN

Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass. 02139, USA

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Under a radiation power constraint, the maximum signal-to-quantum noise ratio obtainable for any state of a radiation field is found. This maximum value is achieved by the two-photon coherent states introduced previously to describe two-photon lasers.

Let the density operator ρ be an arbitrary quantum state of a radiation mode of frequency ω with photon annihilation operator a. We write

$$a = a_1 + ia_2; \quad a_1 = a_1^\dagger, \quad a_2 = a_2^\dagger \tag{1}$$

for selfadjoint a_1, a_2. The signal-to-quantum noise ratio defined by

$$(S/N)_\rho \equiv (\mathrm{tr}\, \rho a_1)^2/\mathrm{tr}\, \rho(\Delta a_1)^2 \tag{2}$$

with $\Delta a_1 \equiv a_1 - \mathrm{tr}\, \rho a_1$ is a useful measure of the intrinsic quantum fluctuation in many optical experiments. In particular, if the operator a_1 is measured at a receiver (optical homodyne) with the transmitted signal in state ρ, $(S/N)_\rho$ provides a measure on how well the signal level $\mathrm{tr}\, \rho a_1$ can be estimated from the measurement. We first show that under the energy (or power per unit frequency) constraint

$$\hbar\omega\, \mathrm{tr}\, \rho a^\dagger a \leqslant \hbar\omega N_s \tag{3}$$

for a given N_s, the optimum $(S/N)_\rho$ achievable by any ρ is

$$(S/N)_o = 4N_s(N_s + 1). \tag{4}$$

Note that under (3), $\mathrm{tr}\, \rho(\Delta a_1)^2$ cannot be made arbitrarily small because that requires an arbitrarily large N_s as apparent in the following.

Eq. (4) can be derived by using the uncertainty principle

$$\mathrm{tr}\, \rho(\Delta a_1)^2\, \mathrm{tr}\, \rho(\Delta a_2)^2 \geqslant 1/16. \tag{5}$$

It is easy to show that all the available energy in (3) should be used and that no energy should be spent in $\mathrm{tr}\, \rho a_2$ so that (3) becomes

$$\mathrm{tr}\, \rho(\Delta a_1)^2 + \mathrm{tr}\, \rho(\Delta a_2)^2 + (\mathrm{tr}\, \rho a_1)^2 = N_s + \tfrac{1}{2}. \tag{6}$$

From (5) it is clear that the minimum $\mathrm{tr}\, \rho(\Delta a_2)^2$ required to satisfy the uncertainty principle for a given $\mathrm{tr}\, \rho(\Delta a_1)^2$ is $[16\, \mathrm{tr}\, \rho(\Delta a_1)^2]^{-1}$. Thus we can write (2) in terms of $\mathrm{tr}\, \rho(\Delta a_1)^2$ alone for the purpose of optimization

$$(S/N)_\rho = \{N_s + \tfrac{1}{2} - \mathrm{tr}\, \rho(\Delta a_1)^2 - [16\, \mathrm{tr}\, \rho(\Delta a_1)^2]^{-1}\}/\mathrm{tr}\, \rho(\Delta a_1)^2. \tag{7}$$

The maximum value of (7) is readily found to be (4) with corresponding

$$\mathrm{tr}\, \rho a_1 = [N_s(N_s + 1)/(2N_s + 1)]^{1/2} \tag{8}$$

$$\mathrm{tr}\, \rho(\Delta a_1)^2 = [4(2N_s + 1)]^{-1}. \tag{9}$$

This optimum value is achieved by the two-photon coherent states $|\beta\rangle_g$ previously referred as "generalized coherent states" [1]. These $|\beta\rangle_g$ are eigenstates of $b \equiv \mu a + \nu a^\dagger$ with eigenvalues β, and $|\mu|^2 - |\nu|^2 = 1$. When $\nu = 0$, $|\beta\rangle_g$ become the ordinary coherent states [2]. With

$$\beta = [N_s(N_s + 1)]^{1/2} \tag{10}$$

$$\mu = (N_s + 1)/(2N_s + 1)^{1/2}, \quad \nu = N_s/(2N_s + 1)^{1/2} \tag{11}$$

one finds from ref. [1] that ${}_g\langle\beta|a_1|\beta\rangle_g$ and ${}_g\langle\beta|(\Delta a_1)^2|\beta\rangle_g$ are indeed given by (8)–(9).

The state $|\beta\rangle_g$ with parameters (10)–(11) is an ordinary minimum uncertainty state. In general, an arbitrary two-photon coherent state $|\beta\rangle_g$ minimizes the

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ratio $(\mathrm{tr}\,\rho a_1')^2/\mathrm{tr}\,\rho(\Delta a_1')^2$ for

$$a_1' = a_1 \cos\phi + a_2 \sin\phi \tag{12}$$

$$\phi = \arctan \frac{i(\mu^*\nu - \nu^*\mu)}{2|\mu||\nu| + \mu^*\nu + \nu^*\mu}. \tag{13}$$

The maximum value (4) obtained from $|\beta\rangle_g$ can be compared to the value

$$(S/N)_{|\alpha\rangle} = 4N_s \tag{14}$$

obtained with a coherent state $\rho = |\alpha\rangle\langle\alpha|$. By spending a portion of available energy as quantum noise in a_2 to decrease the quantum noise in a_1, the resulting signal-to-quantum noise ratio is increased from a coherent state as if N_s is effectively increased to $N_s^2 + N_s$. Among other things, this improvement in $(S/N)_\rho$ leads to a higher information capacity for optical communications with states $|\beta\rangle_g$ [3].

The two-photon coherent states $|\beta\rangle_g$ can be produced from ideal two-photon lasers [1, 4] or degenerate parametric amplifiers [3–6]. Not only do they have absolutely small $\langle\Delta a_1^2\rangle$ discussed previously [1, 4], they also lead to higher (S/N) for a fixed radiation power in the state as shown above. If these $|\beta\rangle_g$ can be experimentally generated from two-photon stimulated processes, they may lead to significant improvement in optical system performance. They may also find applications in physical experiments where a (S/N) beyond the one-photon laser limit (14) is sought for low noise sensitivity

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Two-photon coherent states of the radiation field*

Horace P. Yuen

Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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The concept of a two-photon coherent state is introduced for applications in quantum optics. It is a simple generalization of the well-known minimum-uncertainty wave packets. The detailed properties of two-photon coherent states are developed and distinguished from ordinary coherent states. These two-photon coherent states are mathematically generated from coherent states through unitary operators associated with quadratic Hamiltonians. Physically they are the radiation states of ideal two-photon lasers operating far above threshold, according to the self-consistent-field approximation. The mean-square quantum noise behavior of these states, which is basically the same as those of minimum-uncertainty states, leads to applications not obtainable from coherent states or one-photon lasers. The essential behavior of two-photon coherent states is unchanged by small losses in the system. The counting rates or distributions these states generate in photocount experiments also reveal their difference from coherent states.

I. INTRODUCTION

In a recent study[1] of quantum communication theory it was found that if optical radiations can be generated in a minimum-uncertainty state[2,3] rather than a coherent state,[4] significant performance improvement may result in an optical communication system. For a fixed radiation mode of frequency ω with photon annihilation operator a, the coherent states $|\alpha\rangle$ ($a|\alpha\rangle=\alpha|\alpha\rangle$) have relatively large quantum fluctuations when $\omega/2\pi \geq 10^{12}$ Hz. Writing $a=a_1+ia_2$ for self-adjoint a_1, a_2, a coherent state $|\alpha\rangle$ gives $\langle\Delta a_1^2\rangle=\langle\Delta a_2^2\rangle=\frac{1}{4}$. A noise energy[5] $\frac{1}{4}\hbar\omega$ is then obtained if either a_1 or a_2 is measured in optical homodyne detection. This quantum noise is frequently dominant over other noise sources in optical communications; for example, an equivalent noise temperature of $\simeq 3400\,^\circ$K is obtained for $\frac{1}{4}\hbar\omega$ at the YAlG laser frequency. However, states with $\langle\Delta a_1^2\rangle \ll \frac{1}{4}$ and correspondingly larger $\langle\Delta a_2^2\rangle$ are permitted by the uncertainty principle $\langle\Delta a_1^2\rangle\langle\Delta a_2^2\rangle \geq \frac{1}{16}$, such as the minimum uncertainty states with $\langle\Delta a_1^2\rangle\langle\Delta a_2^2\rangle=\frac{1}{16}$. If one can generate such small $\langle\Delta a_1^2\rangle$ states by an explicit physical process, it should not be surprising that they may be profitably used in many applications.

Minimum-uncertainty states cannot be obtained, however, from available optical sources, all of which generate coherent states and their random or classical superpositions. An investigation of the possible ways to generate minimum-uncertainty states leads to a broad class of radiation states that we call "two-photon coherent states" (TCS's). They include the minimum-uncertainty states of which the coherent state is a special case. A detailed study of these TCS's is reported in this article. A few of our results have been briefly stated in Ref. 6.

In general, a two-photon coherent state differs from a coherent state in several ways: they are generated by different photon processes, they have different quantum statistical properties, and they have different coherence properties. Basically, coherent states are generated from ideal one-photon stimulated processes, whereas TCS's are obtained from ideal stimulated two-photon processes for two photons of the *same* mode. The usual multiphoton parametric processes[7] are one-photon processes in the present sense, as they involve a single photon from each separate mode. The quantum noise properties of TCS's are basically the same as those of minimum-uncertainty states.

Minimum-uncertainty states and coherent states can be easily confused. The above illustration in terms of $\langle\Delta a_1^2\rangle$ and $\langle\Delta a_2^2\rangle$ is a good indication of their difference, if we note that they describe the fluctuations of the field variables directly measured in a homodyne experiment. Further careful discussions will be found in Sec. III E and elsewhere in the paper.

Minimum-uncertainty states were studied at the very early days of quantum mechanics; the Kennard packets[8] are particularly familiar in the context of electrons. It is therefore no surprise that much previous work has touched on various aspects relating to the present two-photon coherent states, including much work on quantum field theory and interacting Bose fluids. It is impossible for the present author to give an accurate historical survey of the literature pertaining to minimum-uncertainty states and their generation by quadratic Hamiltonians. The particularly relevant papers now known to the author include Refs. 9–13 in quantum optics and Refs. 14–17 in quantum field theory. Nevertheless, it appears that TCS's have not been systematically studied before, at least in

the context of quantum optics. These and other additional papers will be mentioned in relation to some of our results whenever appropriate.

This work is not at all a study of possible mathematical generalizations of coherent states. In fact, the TCS's can be considered as "coherent states" in a broad sense, as discussed later. However, they should be distinguished from the coherent states of Glauber,[4] because in quantum optics the term coherent state is usually understood in the Glauber sense which carries the implication of equal fluctuations in the quadratures a_1 and a_2.

In Sec. II we establish notations and briefly recapitulate the basic results in quantum optics used in this paper. For a single degree of freedom, the definition and properties of two-photon coherent states are developed in Sec. III, together with a discussion of the contrast between TCS's and ordinary coherent states. The generation of TCS's from quadratic Hamiltonians is presented in Sec. IV, with illustrative examples. The general structure and properties of the corresponding unitary transformation are fully determined. Physical interpretation of quadratic Hamiltonians is developed in Sec. V in terms of stimulated two-photon processes, which include two-photon lasers and degenerate parametric amplifiers. We argue that TCS's describe ideal two-photon lasing states in analogy to the coherent-state description of one-photon lasers. The approximation which leads to this conclusion is formalized in the Appendix. We also show that the mean-square noise behavior of TCS's remains essentially the same when the effect of small damping and additive noise are included. In Sec. VI we briefly discuss the coherence properties and counting statistics of two-photon coherent states. The usefulness of TCS's in certain applications is brought out in Sec. VII. The detailed theory of multimode TCS's will be given in a separate treatment.

II. COHERENT STATES AND OPERATOR REPRESENTATION

For a single mode of the field with photon annihilation operator a, a coherent state[18,19] $|\alpha\rangle$ is an eigenstate of a,

$$a|\alpha\rangle=\alpha|\alpha\rangle . \tag{2.1}$$

It can be obtained from the vacuum state $|0\rangle$ via the unitary displacement operator $D(\alpha)\equiv e^{\alpha a^\dagger-\alpha^* a}$,

$$|\alpha\rangle=D(\alpha)|0\rangle . \tag{2.2}$$

From (2.2),

$$\langle n|\alpha\rangle=\alpha^n(n!)^{-1/2}e^{-|\alpha|^2/2} . \tag{2.3}$$

If the mode has a harmonic time variation with frequency ω, we have the usual canonical variables (q,p),

$$q\equiv(\hbar/2\omega)^{1/2}(a^\dagger+a),\quad p\equiv i(\tfrac{1}{2}\hbar\omega)^{1/2}(a^\dagger-a) . \tag{2.4}$$

In general, we can introduce two dimensionless self-adjoint operators (a_1,a_2),

$$a\equiv a_1+ia_2;\quad a_1=a_1^\dagger,\quad a_2=a_2^\dagger . \tag{2.5}$$

We also write

$$\alpha\equiv\alpha_1+i\alpha_2;\quad \alpha_1,\ \alpha_2 \text{ real}, \tag{2.6}$$

and refer to (a_1,a_2) or (α_1,α_2) as the *quadrature* components (quantum or classical) of the radiation mode. These quadrature components are the natural variables in describing a TCS, even when the mode has a single frequency. Defining the eigenstates $|\alpha_1\rangle_1$ of a_1, $|\alpha_2\rangle_2$ of a_2,

$$a_1|\alpha_1\rangle_1=\alpha_1|\alpha_1\rangle_1,\quad a_2|\alpha_2\rangle_2=\alpha_2|\alpha_2\rangle_2 , \tag{2.7}$$

we have

$${}_1\langle\alpha_1|\alpha'\rangle=(2/\pi)^{1/4}\exp[-(\alpha_1-\alpha')^2+\tfrac{1}{2}\alpha'(\alpha'-\alpha'^*)] . \tag{2.8}$$

From (2.6) it follows easily that

$$\langle a\rangle=\alpha,\quad \langle N\rangle=|\alpha|^2 , \tag{2.9}$$

$$\langle\Delta a_1^2\rangle=\langle\Delta a_2^2\rangle=\tfrac{1}{4} , \tag{2.10}$$

where $\langle\Delta M^2\rangle\equiv\langle(M-\langle M\rangle)^2\rangle$ for an operator M. The angular bracket $\langle M\rangle$ represents the expected value $\mathrm{tr}\rho M$ of M with respect to a quantum state ρ which may be either pure or mixed. The uncertainty principle on (a_1,a_2) is

$$\langle\Delta a_1^2\rangle\langle\Delta a_2^2\rangle\geq\tfrac{1}{16} . \tag{2.11}$$

Thus $|\alpha\rangle$ is a minimum-uncertainty state. From the arithmetic-geometric mean inequality, (2.11) implies

$$\langle\Delta a_1^2\rangle+\langle\Delta a_2^2\rangle\geq\tfrac{1}{2} , \tag{2.12}$$

so that $|\alpha\rangle$ can be characterized as the states with both minimum-uncertainty product and minimum-uncertainty sum.

The differential operator representation of $M(a^\dagger,a)$ in the coherent-state representation is[20,19]

$$\langle\alpha|M(a^\dagger,a)|\psi\rangle=M\left(\alpha^*,\frac{\alpha}{2}+\frac{\partial}{\partial\alpha^*}\right)\langle\alpha|\psi\rangle , \tag{2.13}$$

from which one can show

$$\langle\alpha|M(a^\dagger,a)|\beta\rangle=\langle\alpha|\beta\rangle M^{(n)}(\alpha^*,\beta) , \tag{2.14}$$

where

$$M^{(n)}(\alpha^*,\alpha)\equiv\langle\alpha|M|\alpha\rangle \tag{2.15}$$

is the normal-order form[3] of M. Note that it is important to keep the separate dependence of $M^{(n)}$

on α^* and α in (2.14)—$M^{(n)}$ is a function of two complex variables (α^*, α). Equation (2.14) explicitly shows the fact[19] that $M^{(n)}(\alpha^*, \alpha)$ uniquely determines M. In terms of $U^{(n)}(\alpha^*, \alpha; t, t_0)$, the Schrödinger equation $i\hbar\,\partial U/\partial t = H(a^\dagger, a)U$ becomes[19]

$$i\hbar\frac{\partial U^{(n)}}{\partial t} = H\left(\alpha^*, \alpha + \frac{\partial}{\partial\alpha^*}\right)U^{(n)}. \tag{2.16}$$

These results are at least valid for bounded operators or unbounded operators which are polynomials in a and $a^\dagger$.

In a number of mathematical senses,[21-24] any density operator ρ can be expressed in the diagonal P representation,[4,25]

$$\rho = \int P(\alpha^*, \alpha)\,|\alpha\rangle\langle\alpha|\,d^2\alpha, \quad d^2\alpha \equiv d(\mathrm{Re}\,\alpha)\,d(\mathrm{Im}\,\alpha). \tag{2.17}$$

Let $\chi_N(\eta^*, \eta)$ be the normally ordered characteristic function

$$\chi_N(\eta^*, \eta) \equiv \mathrm{tr}(\rho e^{\eta a^\dagger} e^{-\eta^* a}), \tag{2.18}$$

then $P(\alpha^*, \alpha)$ is the two-dimensional Fourier transform of $\chi_N(\eta^*, \eta)$,[26,21]

$$P(\alpha^*, \alpha) = \int e^{\eta^*\alpha - \alpha^*\eta}\chi_N(\eta^*, \eta)\,d^2\eta/\pi^2. \tag{2.19}$$

It has not been demonstrated that $P(\alpha^*, \alpha)$ provides a convenient tool in actual calculation when it is not sufficiently well behaved.

Being the trace of the product of a bounded operator and a trace-class operator, the characteristic functions $\chi_N(\eta^*, \eta)$, $\chi_A(\eta^*, \eta)$, and $\chi_W(\eta^*, \eta)$ are always well defined,

$$\chi_A(\eta^*, \eta) \equiv \mathrm{tr}(\rho e^{-\eta^* a} e^{\eta a^\dagger}), \tag{2.20}$$

$$\chi_W(\eta^*, \eta) \equiv \mathrm{tr}(\rho e^{-\eta^* a + \eta a^\dagger}). \tag{2.21}$$

The Fourier transform of $\chi_W(\eta^*, \eta)$ is the Wigner distribution and that of $\chi_A(\eta^*, \eta)$ is $\rho^{(n)}(\alpha^*, \alpha)$,[27,21] $\rho^{(n)}(\alpha^*, \alpha) \equiv \langle\alpha|\rho|\alpha\rangle$,

$$W(\alpha^*, \alpha) = \int \chi_W(\eta^*, \eta)e^{\eta^*\alpha - \eta\alpha^*}\,d^2\eta/\pi^2, \tag{2.22}$$

$$\chi_A(\eta^*, \eta) = \int \rho^{(n)}(\alpha^*, \alpha)e^{\eta\alpha^* - \alpha\eta^*}\,d^2\alpha/\pi. \tag{2.23}$$

The characteristic functions are related by

$$\chi_A(\eta^*, \eta) = e^{-|\eta|^2}\chi_N(\eta^*, \eta) = e^{-|\eta|^2/2}\chi_W(\eta^*, \eta), \tag{2.24}$$

which also implies relations between $P(\alpha^*, \alpha)$, $\rho^{(n)}(\alpha^*, \alpha)$, and $W(\alpha^*, \alpha)$.

The function $\rho^{(n)}(\alpha^*, \alpha)$ is always well behaved, in contrast to $P(\alpha^*, \alpha)$, and is also quite convenient for moment calculations. Furthermore, it specifies ρ uniquely as in (2.14) and is the probability density function[28] describing the outcome statistics of a simultaneous quantum measurement of two noncommuting observables[29,30] (or a measurement of the non-self-adjoint operator a, optical heterodyning).[31] We will use $\rho^{(n)}(\alpha^*, \alpha)$ or the characteristic functions to represent a density operator, particularly because $P(\alpha^*, \alpha)$ for a TCS is too singular. In terms of $\rho^{(n)}$, we have

$$\rho = \int e^{\eta a^* - a\eta^*}\rho^{(n)}(\alpha^*, \alpha)e^{-\eta a^\dagger}e^{\eta^* a}\,d^2\alpha\,d^2\eta/\pi^2, \tag{2.25}$$

$$\mathrm{tr}\rho M = \int [\chi_A^M(\eta^*, \eta)]^*\chi_A^\rho(\eta^*, \eta)e^{|\eta|^2}\,d^2\eta/\pi, \tag{2.26}$$

if $\chi_A^M(\eta^*, \eta) = \mathrm{tr}(Me^{-\eta^* a}e^{\eta a^\dagger})$ exists.

III. TWO-PHOTON COHERENT STATES

A. Definition

For a fixed radiation mode with photon annihilation operator a, let

$$b \equiv \mu a + \nu a^\dagger \tag{3.1}$$

for a pair of c numbers μ, ν obeying

$$|\mu|^2 - |\nu|^2 = 1. \tag{3.2}$$

It follows from (3.2) that

$$[b, b^\dagger] = 1. \tag{3.3}$$

Any transformation $b(a^\dagger, a)$ which leaves the commutator invariant as in (3.3) is called a canonical transformation. The change of variables from $(a, a^\dagger)$ to $(b, b^\dagger)$ according to (3.1) and (3.2) is therefore a linear canonical transformation. A theorem of Von Neumann[32,17] asserts that every canonical transformation can be represented as a unitary transformation, i.e.,

$$b(a^\dagger, a) = UaU^\dagger \tag{3.4}$$

for a unitary operator U. The structure and physical realization of the U_L that leads to the linear canonical transformation (3.1) are discussed in Secs. IV and V;

$$U_L a U_L^\dagger \equiv \mu a + \nu a^\dagger. \tag{3.5}$$

The commutator (3.3) or Eq. (3.4) provides b with properties exactly similar to those of a. One obtains the following in a way identical to the usual derivation for N and a:[3] Let N_g be the "quasiphoton" number operator,

$$N_g \equiv b^\dagger b = U_L N U_L^\dagger. \tag{3.6}$$

Then N_g has discrete positive eigenvalues n_g with ground state $|0_g\rangle$,

$$N_g|m_g\rangle = m_g|m_g\rangle, \quad N_g|0_g\rangle = 0, \tag{3.7}$$

$$|m_g\rangle = U_L|m\rangle. \tag{3.8}$$

Similar to $|n\rangle$, the states $|m_g\rangle$ can be expressed in the form

$$|m_g\rangle = (b^\dagger)^m (m!)^{-1/2} |0_g\rangle . \tag{3.9}$$

They are complete orthonormal. Furthermore, b acts as the lowering operator for $|m_g\rangle$. The quasiparticle interpretation of b and $|m_g\rangle$ is discussed in Sec. IV A.

The TCS's $|\beta\rangle_g$ are *defined* to be the eigenstates of b with eigenvalues β,

$$b|\beta\rangle_g = \beta|\beta\rangle_g . \tag{3.10}$$

The μ,ν dependence of a TCS has been suppressed in the notation $|\beta\rangle_g$. When necessary, we will use the more complete notation $|\beta;\mu,\nu\rangle$. With $\nu=0$, $|\beta\rangle_g$ become the ordinary coherent states. In our later references to a TCS, it should be implicitly understood that $\nu\neq 0$.

From (3.4), there is an U_L independent of β which gives

$$|\beta\rangle_g = U_L|\beta\rangle \tag{3.11}$$

for each state $|\beta\rangle$. Therefore in parallel with $|\beta\rangle$

$$|\beta\rangle_g = D_g(\beta)|0\rangle_g, \quad D_g(\beta) \equiv e^{\beta b^\dagger - \beta^* b}, \tag{3.12}$$

$$\int |\beta\rangle_{g\,g}\langle\beta|\, d^2\beta/\pi = I , \tag{3.13}$$

$${}_g\langle\beta|\beta'\rangle_g = \exp(\beta^*\beta' - \tfrac{1}{2}|\beta|^2 - \tfrac{1}{2}|\beta'|^2). \tag{3.14}$$

If the notation $|\beta;\mu,\nu\rangle$ is not used, the same μ,ν is understood to apply to all of the $|\beta\rangle_g$ in an equation.

From (3.13), any state vector $|\psi\rangle$ can be expanded in terms of $|\beta\rangle_g$. Similarly, diagonal operator representation can be developed,

$$M = \int M^{(a)}(\beta^*,\beta)|\beta\rangle_{g\,g}\langle\beta|\, d^2\beta/\pi , \tag{3.15}$$

for the antinormal order form $M^{(a)}(\beta^*,\beta)$ of M. In the $|\beta\rangle_g$ representation, we have [similar to (2.21)]

$${}_g\langle\beta|M(b^\dagger,b)|\psi\rangle = M\left(\beta^*, \frac{\beta}{2}+\frac{\partial}{\partial\beta^*}\right){}_g\langle\beta|\psi\rangle . \tag{3.16}$$

Other properties associated with $|\alpha\rangle$ carry over identically to $|\beta\rangle_g$.

The results of this subsection on $|\beta\rangle_g$ and $|m_g\rangle$ depends only on the relation (3.4), and not on the more specific (3.5). The specific characteristics of (3.1) are developed below. Certain properties of TCS's have been discussed by Stoler[9] in the context of minimum-uncertainty states.

B. Wave functions

The general coherent-state wave function $\langle\alpha|\beta\rangle_g$ of a TCS is first determined as follows: From (3.1), (3.10), and (2.13) we obtain

$$\frac{\partial}{\partial\alpha^*}\langle\alpha|\beta\rangle_g = \left(\frac{\beta}{\mu}-\frac{\alpha}{2}-\frac{\nu\alpha^*}{\mu}\right)\langle\alpha|\beta\rangle_g . \tag{3.17}$$

The solution of (3.17) is of the form

$$\langle\alpha|\beta\rangle_g = \exp[\alpha^*\beta/\mu - \tfrac{1}{2}|\alpha|^2 - \nu\alpha^{*2}/2\mu + f(\beta^*,\beta)] . \tag{3.18}$$

The (α^*,α)-independent f can be determined from (3.16), (3.1), (2.13), and

$${}_g\langle\beta|b|\alpha\rangle = \left(\frac{\beta}{2}+\frac{\partial}{\partial\beta^*}\right){}_g\langle\beta|\alpha\rangle$$

$$= \left(\mu\alpha + \frac{\nu\alpha^*}{2} + \nu\frac{\partial}{\partial\alpha}\right){}_g\langle\beta|\alpha\rangle , \tag{3.19}$$

so that the β-dependent part of f in (3.18) is $-\tfrac{1}{2}|\beta|^2 + \nu^*\beta^2/2\mu$. From the normalization

$$\int |\langle\alpha|\beta\rangle_g|^2 d^2\alpha/\pi = 1 ,$$

$\langle\alpha|\beta\rangle_g$ is thus determined up to a $(\alpha^*,\alpha;\beta^*,\beta)$-independent phase in the form

$$\langle\alpha|\beta\rangle_g = \mu^{-1/2}\exp[-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2 - (\nu/2\mu)\alpha^{*2} + (\nu^*/2\mu)\beta^2 + (1/\mu)\alpha^*\beta + i\theta_0] \tag{3.20}$$

for a real θ_0. The normalization constant is calculated from the normalization of Gaussian integrals.[20] For $\mu=1$, $\nu=0$, (3.20) reduces to the usual form of $\langle\alpha|\beta\rangle$ for $\theta_0=0$, a choice of phase we will adopt for $\langle\alpha|\beta\rangle_g$.

While we have

$$\langle m_g|\beta\rangle_g = \beta^m(m!)^{-1/2}e^{-|\beta|^2/2} , \tag{3.21}$$

similar to (2.3), the function $\langle m|\beta\rangle_g$ takes a more complicated form which can be determined as follows: As a function of the complex variables z and t, e^{2zt-t^2} can be expanded in the Taylor series

$$e^{2zt-t^2} = \sum_{n=0}^{\infty}\frac{H_n(z)t^n}{n!}, \quad |t|<\infty , \tag{3.22}$$

where $H_n(z)$ is the nth Hermite polynomial with complex argument z. Writing

$$\langle\alpha|\beta\rangle_g = \sum_n \langle\alpha|n\rangle\langle n|\beta\rangle_g$$

and using (3.20) and (2.3), we find

$$\langle n|\beta\rangle_g = (n!\,\mu)^{-1/2}(\nu/2\mu)^{n/2}H_n[\beta(2\mu\nu)^{-1/2}] \times \exp[-\tfrac{1}{2}|\beta|^2 + (\nu^*/2\mu)\beta^2] . \tag{3.23}$$

For $\mu=1$, $\nu=0$, the asymptotic forms of $H_n(z)$ for various[33] arg$z$ all lead to (2.3), as they must. Equation (3.23) shows that the counting statistics of a TCS is far from the Poisson (2.3).

The wave function ${}_1\langle\alpha_1|\beta\rangle_g$ can be readily determined from (3.20) and (2.8). Thus we find

$$ {}_1\langle\alpha_1|\beta\rangle_g=\left(\frac{2}{\pi}\right)^{1/4}(\mu-\nu)^{-1/2}\exp\left(-\tfrac{1}{2}|\beta|^2-\frac{\mu^*-\nu^*}{2(\mu-\nu)}\beta^2+2\alpha_1\frac{\beta}{\mu-\nu}-\frac{\mu+\nu}{\mu-\nu}\alpha_1^2\right). \tag{3.24}$$

From (3.20) we have

$$\langle\beta;\mu,\nu|\beta_0;\mu_0,\nu_0\rangle=(\mu_0\mu^*-\nu_0\nu^*)^{-1/2}$$
$$\times\exp\left(-\tfrac{1}{2}|\beta|^2-\tfrac{1}{2}|\beta_0|^2+\frac{1}{\mu_0\mu^*-\nu_0\nu^*}\beta^*\beta_0-\frac{\nu_0\mu-\mu_0\nu}{2(\mu_0\mu^*-\nu_0\nu^*)}\beta^{*2}+\frac{(\nu_0\mu-\mu_0\nu)^*}{2(\mu_0\mu^*-\nu_0\nu^*)}\beta_0^2\right). \tag{3.25}$$

Note the Gaussian form of (3.24) and (3.25). Various reduction of (3.24) and (3.25) to the previous formulas can be easily observed. The relative phases of (3.20), (3.21), (3.24), and (3.25) are mutually consistent, as well as being consistent with the above wave functions for the coherent states.

From

$$\langle n|\beta\rangle_g=\sum_m\langle n|m_g\rangle\langle m_g|\beta\rangle_g$$

for the quasiphoton number states $|m_g\rangle$ we obtain

$$\langle n|m_g\rangle=Q_1(\mu,\nu;m,n),\qquad m\le n,\quad m\text{ odd},\quad n\text{ odd}, \tag{3.26a}$$

$$=Q_1(\mu,\nu;m,n)+Q_2(\mu,\nu;m,n),\qquad m\le n,\quad m\text{ even},\quad n\text{ even}, \tag{3.26b}$$

$$=Q_2(\mu,\nu;m,n),\qquad m>n,\quad m\text{ even},\quad n\text{ even}, \tag{3.26c}$$

$$=0,\quad\text{otherwise}; \tag{3.26d}$$

$$Q_1(\mu,\nu;m,n)\equiv\mu^{-(m+n+1)/2}\nu^{(n-m)/2}\frac{(-2)^{(m-n)/2}(n!)^{1/2}}{[(n-m)/2]!\,(m!)^{1/2}}, \tag{3.26e}$$

$$Q_2(\mu,\nu;m,n)$$
$$\equiv\mu^{-(m+n+1)/2}\nu^{n/2}\nu^{*m/2}(-1)^n2^{-(n+m)/2}\frac{(n!\,m!)^{1/2}}{(n/2)!\,(m/2)!}. \tag{3.26f}$$

Note that m and n must be both even or both odd in order that $\langle n|m_g\rangle$ be nonvanishing. Thus only an even number of photons would be counted for the state $|0_g\rangle=|0\rangle_g$. Other wave functions of $|m_g\rangle$ can be obtained accordingly. These wave functions provide the statistics of various measurements on the radiation states $|\beta\rangle_g$ and $|m_g\rangle$.

C. Quantum fluctuations and characteristic functions

From (3.1) and (3.2),

$$a=\mu^*b-\nu b^\dagger, \tag{3.27}$$

so that for the TCS $|\beta\rangle_g$,

$$\langle a\rangle\equiv{}_g\langle\beta|a|\beta\rangle_g=\mu^*\beta-\nu\beta^*\equiv\hat\beta\equiv\hat\beta_1+i\hat\beta_2, \tag{3.28}$$

$$\langle a^\dagger a\rangle=|\hat\beta|^2+|\nu|^2, \tag{3.29}$$

$$\langle\Delta a^2\rangle=-\mu^*\nu=\langle\Delta a^{\dagger2}\rangle^*, \tag{3.30}$$

$$\langle\Delta a_1^2\rangle=\tfrac{1}{4}|\mu-\nu|^2, \tag{3.31a}$$

$$\langle\Delta a_2^2\rangle=\tfrac{1}{4}|\mu+\nu|^2, \tag{3.31b}$$

$$\langle\Delta a_1\Delta a_2\rangle\equiv\langle(a_1-\hat\beta_1)(a_2-\hat\beta_2)\rangle=\tfrac{1}{4}i(\mu^*\nu-\nu^*\mu+1), \tag{3.32a}$$

$$\langle\Delta a_2\Delta a_1\rangle=\tfrac{1}{4}i(\mu^*\nu-\nu^*\mu-1). \tag{3.32b}$$

Equation (3.32) is, of course, consistent with the commutator

$$[a_1,a_2]=[b_1,b_2]=\tfrac{1}{2}i. \tag{3.33}$$

Compared to the case of a coherent state where

$$\langle\alpha|(\Delta a)^\dagger(\Delta a)|\alpha\rangle=0, \tag{3.34}$$

a TCS $|\beta\rangle_g$ has added quantum noise energy

$${}_g\langle\beta|(\Delta a)^\dagger(\Delta a)|\beta\rangle_g=|\nu|^2. \tag{3.35}$$

For self-adjoint b_1, b_2, and real β_1, β_2, let

$$b\equiv b_1+ib_2,\qquad\beta\equiv\beta_1+i\beta_2. \tag{3.36}$$

The TCS $|\beta\rangle_g$ can be characterized as the states which minimize both $\langle\Delta b_1^2\rangle\langle\Delta b_2^2\rangle\ge\frac{1}{16}$ and $\langle\Delta b_1^2\rangle+\langle\Delta b_2^2\rangle\ge\frac{1}{2}$. On the other hand, $|\beta\rangle_g$ minimizes the uncertainty product *only* when

$$\mu=\delta\nu,\quad\text{for }\delta\text{ real}. \tag{3.37}$$

These minimum uncertainty states[2] include the eigenstates of a_1, a_2 in the limit $\mu,\nu\to\infty$, such that (3.2) holds. Specifically, the eigenstates $|\alpha_1\rangle_1$ are obtained when $\delta\to1$ and the eigenstates $|\alpha_2\rangle_2$ are obtained when $\delta\to-1$. As with the minimum-uncertainty states, the mean-square quantum fluctuations in a_1 and a_2 can be exchanged in $|\beta\rangle_g$ from (3.31). On the other hand, they are fixed at $\frac{1}{4}$ for coherent states.

Thus for a mode of frequency ω, a significant noise energy[5] $\frac{1}{4}\hbar\omega$ is obtained if either of the quadrature components is measured in a coherent state, while $\langle\Delta a_1^2\rangle\ll\frac{1}{4}$ with corresponding large $\langle\Delta a_2^2\rangle$ can be obtained for measurement of a_1 in a TCS.

A mathematical scaling of a will, of course, lead to $\langle\Delta a_1^2\rangle < \frac{1}{4}$ even in an ordinary coherent state. For example, $\langle\Delta b_1^2\rangle < \frac{1}{4}$ is possible with b given by (3.1). *In actual applications such scaling cannot* affect the system performance. In Sec. VII B we will show how the noise in a coherent signal can be reduced if $|\beta\rangle_g$ is available, which is impossible by scaling on $|\beta\rangle$. The fact is that $\langle\Delta a_1^2\rangle$ measures the absolute amount of noise in a quadrature component and a is a more fundamental variable than b. See Secs. III E, IV A, and VII for further discussion and clarification.

From (3.20) one readily obtains

$$|\langle\alpha|\beta\rangle_g|^2 = |\mu|^{-1}\exp[-(1-2C_1)(\alpha_1-\hat\beta_1)^2 - (1+2C_1)(\alpha_2-\hat\beta_2)^2 + 4C_2(\alpha_1-\hat\beta_1)(\alpha_2-\hat\beta_2)], \tag{3.38}$$

where

$$C \equiv -\nu/2\mu \equiv C_1 + iC_2, \quad C_1,\ C_2 \text{ real}, \tag{3.39}$$

and $\hat\beta$ is given by (3.28). From (3.2), $|C|^2 \leq \frac{1}{4}$, so that

$$|C_1| \leq \tfrac{1}{2}, \quad |C_2| \leq \tfrac{1}{2}. \tag{3.40}$$

Thus (3.38) is indeed a Gaussian joint probability density function in (α_1, α_2), with mean $(\hat\beta_1, \hat\beta_2)$ and variance

$$\sigma_1^2 \equiv E[(\alpha_1-\hat\beta_1)^2] = \tfrac{1}{2}(1+2C_1)/(1-4|C|^2), \tag{3.41a}$$

$$\sigma_2^2 \equiv E[(\alpha_2-\hat\beta_2)^2] = \tfrac{1}{2}(1-2C_1)/(1-4|C|^2). \tag{3.41b}$$

The notation $E[R]$ indicates the probabilistic average of a random variable R with respect to its density function.[31] The correlation coefficient is

$$r \equiv E[(\alpha_1-\hat\beta_1)(\alpha_2-\hat\beta_2)]/\sigma_1\sigma_2 = 2C_2(1-4C_1)^{-1/2}. \tag{3.42}$$

This density function is illustrated in Fig. 1.

Equation (3.38) is conveniently expressed in the matrix form

$$\rho_g^{(n)}(\alpha_1, \alpha_2) \equiv |\langle\alpha|\beta\rangle_g|^2 = |\mu|^{-1}\exp[-\tfrac{1}{2}(\underline{\alpha}_q - \underline{\bar\alpha}_q)^T\underline{\Lambda}^{-1}(\underline{\alpha}_q - \underline{\bar\alpha}_q)], \tag{3.43}$$

where

$$\underline{\alpha}_q \equiv \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}, \quad \underline{\alpha}_q^T \equiv (\alpha_1, \alpha_2), \tag{3.44}$$

$$\underline{\bar\alpha}_q \equiv E[\underline{\alpha}_q] = \underline{\hat\beta}_q, \tag{3.45}$$

$$\underline{\Lambda} \equiv \underline{\Lambda}_s + \tfrac{1}{4}\underline{I}, \quad \underline{I} \equiv \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{3.46}$$

$$\underline{\Lambda}_s \equiv \begin{pmatrix}\langle\Delta a_1^2\rangle & \tfrac{1}{2}\langle\Delta a_1\Delta a_2 + \Delta a_2\Delta a_1\rangle\\ \tfrac{1}{2}\langle\Delta a_1\Delta a_2 + \Delta a_2\Delta a_1\rangle & \langle\Delta a_2^2\rangle\end{pmatrix}, \tag{3.47}$$

$$\underline{\Lambda}^{-1} = 2\begin{pmatrix}1-2C_1 & -2C_2\\ -2C_2 & 1+2C_1\end{pmatrix}. \tag{3.48}$$

In terms of the variables (α^*, α), (3.38) becomes

$$\rho_g^{(n)}(\alpha^*, \alpha) = |\mu|^{-1}\exp[-\tfrac{1}{2}(\underline{\alpha}_a - \underline{\bar\alpha}_a)^T\underline{\Lambda}^{-1}(\underline{\alpha}_a - \underline{\bar\alpha}_a)] \tag{3.49}$$

through the transformation

$$\underline{\alpha}_a \equiv \begin{pmatrix}\alpha\\ \alpha^*\end{pmatrix} = \begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \equiv \underline{\Omega}\,\underline{\alpha}_q, \tag{3.50}$$

$$\underline{\Lambda}_a \equiv \begin{pmatrix}\langle\Delta a^2\rangle & \langle\Delta a\Delta a^\dagger\rangle\\ \langle\Delta a\Delta a^\dagger\rangle & \langle\Delta a^{\dagger 2}\rangle\end{pmatrix} = \underline{\Omega}\,\underline{\Lambda}\,\underline{\Omega}^T, \tag{3.51}$$

where T denotes the transpose of a matrix.

The antinormally ordered characteristic function $\chi_A^g(\eta^*, \eta)$ for

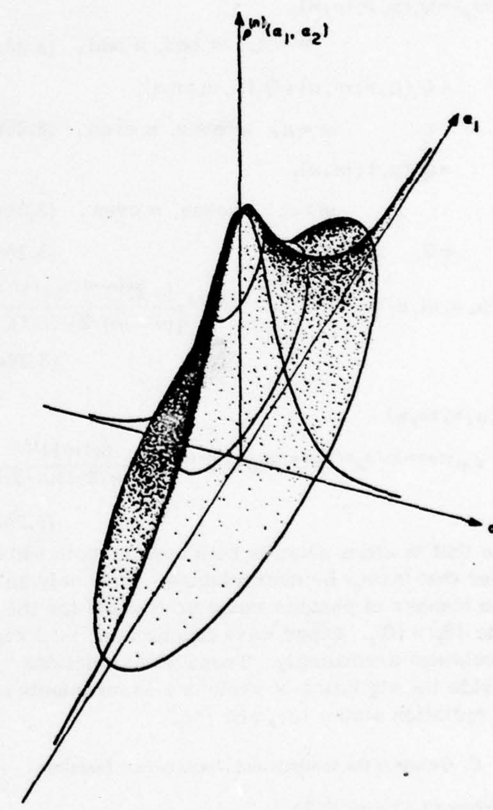

FIG. 1. Representation of a two-photon coherent state by $\rho^{(n)}(\alpha_1, \alpha_2)$ from (3.38), with $\sigma_1^2 > \sigma_2^2$; a coherent state is obtained for $\sigma_1^2 = \sigma_2^2$.

$$\rho_g \equiv |\beta\rangle_{g\,g}\langle\beta| \tag{3.52}$$

is then

$$\chi_A^g(\eta_1,\eta_2) = \exp(i\underline{\eta}_q^T\underline{\overline{\alpha}}_q - \tfrac{1}{2}\underline{\eta}_q^T\underline{\Lambda}\,\underline{\eta}_q)\,, \tag{3.53}$$

$$\underline{\eta}_q \equiv \begin{pmatrix} 2\eta_2 \\ -2\eta_1 \end{pmatrix}, \quad \eta \equiv \eta_1 + i\eta_2\,, \tag{3.54}$$

and the Weyl characteristic function is, from (2.32),

$$\chi_W^g(\eta_1,\eta_2) = \exp(i\underline{\eta}_q^T\underline{\overline{\alpha}}_q - \tfrac{1}{2}\underline{\eta}_q^T\underline{\Lambda}_s\underline{\eta}_q)\,. \tag{3.55}$$

Since $\underline{\Lambda} > 0$ (positive definite),[34] the Wigner distribution is also a Gaussian density function. The antinormally ordered characteristic function $\chi_N^g(\eta_1,\eta_2)$ is

$$\chi_N^g(\eta_1,\eta_2) = \exp[i\underline{\eta}_q^T\underline{\overline{\alpha}}_q - \tfrac{1}{2}\underline{\eta}_q^T(\underline{\Lambda} - \tfrac{1}{4}\underline{I})\underline{\eta}_q]\,. \tag{3.56}$$

The antinormal-, symmetrized-, and normal-order moments of $(a^\dagger, a)$ can be calculated from (3.53), (3.55), and (3.56), respectively. Note that these χ_N, χ_W, χ_A, and W, $\rho^{(n)}$ for TCS's are all Gaussian. The above matrix forms are convenient for generalization to the multimode situations.

The minimum noise $\langle\Delta b_1^2\rangle$ obtainable in a TCS for a self-adjoint b_1,

$$b_1 \equiv \gamma_1 a_1 + \gamma_2 a_2\,, \quad \gamma_1^2 + \gamma_2^2 = 1\,, \tag{3.57}$$

is the minimum eigenvalue of $\underline{\Lambda}_s$, with the minimizing value of (γ_1,γ_2) given by the corresponding eigenvector.[35] The result can be expressed in the following form: Let

$$\gamma_1 + i\gamma_2 \equiv e^{-i\phi} \tag{3.58}$$

for a real ϕ and let

$$a' \equiv a e^{i\phi}\,. \tag{3.59}$$

Then a_1' minimizes $\langle\Delta b_1^2\rangle$ when

$$\phi = \tan^{-1}[i(\mu^*\nu - \nu^*\mu)/(2|\mu|\,|\nu| + \mu^*\nu + \nu^*\mu)] \tag{3.60}$$

and the minimum fluctuation is

$$\langle\Delta a_1'^2\rangle = \tfrac{1}{4}(|\mu| - |\nu|)^2\,. \tag{3.61a}$$

The conjugate variable a_2' is the one which maximizes $\langle\Delta b_2^2\rangle$, with

$$\langle\Delta a_2'^2\rangle = \tfrac{1}{4}(|\mu| + |\nu|)^2\,. \tag{3.61b}$$

From (3.61), the TCS $|\beta\rangle_g$ can be seen to minimize the uncertainty product

$$\langle\Delta a_1'^2\rangle\langle\Delta a_2'^2\rangle \geq \tfrac{1}{16}\,, \tag{3.62}$$

for a' defined by (3.59) and (3.60). Thus $|\beta\rangle_g$ are the minimum-uncertainty-product states for a' and can therefore be understood as generalized or rotated minimum-uncertainty states. This fact is important in some applications as indicated in Sec. VII B, because the original $\langle\Delta a_1^2\rangle$ and $\langle\Delta a_2^2\rangle$ can be both larger than $\frac{1}{4}$ from (3.31) unless μ/ν is real.

D. Diagonal operator representation for $|\beta\rangle_{g\,g}\langle\beta|$

Let ρ_+ represent a density operator which can be written as a random superposition of coherent states, i.e.,

$$\rho_+ = \int P_+(\alpha^*,\alpha)\,|\alpha\rangle\langle\alpha|\,d^2\alpha\,, \tag{3.63}$$

for a positive true probability density function $P_+(\alpha^*,\alpha)$. It follows that

$$\mathrm{tr}\rho_+(\Delta a_1^2) = \frac{1}{4} + \int P_+(\alpha_1,\alpha_2)(\alpha_1 - \overline{\alpha}_1)^2\,d\alpha_1\,d\alpha_2 \geq \tfrac{1}{4}\,, \tag{3.64a}$$

for

$$\overline{\alpha}_1 = \int P_+(\alpha_1,\alpha_2)\alpha_1\,d\alpha_1\,d\alpha_2\,,$$

and similarly,

$$\mathrm{tr}\rho_+(\Delta a_2^2) \geq \tfrac{1}{4}\,. \tag{3.64b}$$

Equation (3.64) states the obvious physical fact that the introduction of classical randomness merely increases the noise in a_1 and a_2.

A TCS with (3.31) therefore cannot be a mixture of coherent states. In fact, the density operator ρ_g cannot be written in the diagonal form (2.17) with $P(\alpha^*,\alpha)$ belonging to the class of tempered distribution. This can be seen from the normally ordered characteristic function $\chi_N^g(\eta_1,\eta_2)$. For a TCS, $\underline{\Lambda} - \frac{1}{4}\underline{I}$ cannot be positive definite. Thus χ_N^g does not possess a Fourier transform. If we insist on having a diagonal P representation for ρ_g, we will need the Fourier transform of $e^{\gamma x^2}$ for a real variable x and $\gamma > 0$. It does not appear to be useful to talk about the Fourier transform of such a quantity, which certainly does not belong to the class of tempered distribution. Further discussion of this divergence is given in Ref. 27.

One may still consider the diagonal P representation of ρ_g as belonging to the class of distribution Z_2,[24] or alternatively as the limit of a sequence of well-behaved density functions.[23] Since no useful method of calculation has resulted from such viewpoints and since $\rho^{(n)}(\alpha^*,\alpha)$ is sufficient for our purpose, we will say that the P representation of ρ_g does not exist and we will not employ it in the following.

Note, however, that a generalized P_g representation of ρ_g does exist and can be useful. Thus

$$\rho = \int P_g(\beta^*,\beta)\,|\beta\rangle_{g\,g}\langle\beta|\,d^2\beta/\pi \tag{3.65}$$

may be used to represent $|\overline{\beta}\rangle_g$, with

$$P_g(\beta^*, \beta) = \delta^2(\beta - \bar{\beta}). \qquad (3.66)$$

Equation (3.66) also describes random mixtures of $|\beta\rangle_g$ in the same way as (3.63). In particular, it remains useful with the introduction of an additive noise. However, it is inconvenient if (μ, ν) are time varying. While it can be used in moment calculations by expressing $M(a^\dagger, a)$ in normal-order form $M^{(n)}(b^\dagger, b)$ in terms of the variables $(b^\dagger, b)$, it is no longer particularly suited to be calculations of $(a^\dagger, a)$ normal-order moments encountered in the description of most experimental results. Therefore (3.65) will also not be employed in this paper.

E. Coherent states versus two-photon coherent states

As mentioned in Sec. I, two-photon coherent states were encountered in many different studies, but they received little attention in quantum optics. Obviously this is due largely to the fact that these states are not generated from available optical sources, but other factors may have also contributed.

It is well known that the mathematical form of a minimum-uncertainty state at a frequency ω_1 is identical to that of a coherent state at a different frequency ω_2. Therefore one may tend to view minimum-uncertainty states as just coherent states of a different frequency, particularly since the frequency is often normalized away. This viewpoint is erroneous, because in the state space $\mathcal{H}(\omega_1) \otimes \mathcal{H}(\omega_2)$ a minimum-uncertainty state at ω_1 is given by $|\beta\rangle_g \otimes |0\rangle$, while a coherent state at ω_2 is $|0\rangle \otimes |\beta\rangle$. Physically, no photocounts would be registered for a coherent state at ω_2 in a counting experiment employing detectors responding only to fields at ω_1. In this connection one may also note that for a fixed mode the frequency is also fixed by the dispersion relation. Changing the frequency of a fixed mode by dielectric modulation would not yield a minimum-uncertainty state at frequency ω_1 from a coherent state at frequency ω_1. A coherent state at a different frequency ω_2 would be obtained instead.

Alternatively, one may regard TCS's and ordinary coherent states as (mathematically) equivalent, since they are related through a unitary transformation or some kind of scaling. The question of scaling was already dealt with in Sec. III C concerning the noise in quadrature components. It is true that two-photon coherent states are coherent states in a broad sense: they are the coherent states with respect to $b(\nu \neq 0)$ instead of a. However, this abstract unitary equivalence means little in describing actual experimental results. Thus $\langle n_g | \alpha\rangle_g = \langle n | \alpha\rangle$ but $\langle n | \alpha\rangle_g$ has little resemblance to $\langle n | \alpha\rangle$. In general, for an arbitrary unitary transformation U we can define a set of states $|\ \rangle$,

$$|\ \rangle = U|\alpha\rangle. \qquad (3.67)$$

The functions $\langle x |\ \rangle$ cannot be obtained by simple scaling on $\langle x | \alpha\rangle$ and has to be computed anew via $\langle x | U | \alpha\rangle$. This paper develops such specific results for the U_L of (3.5).

With respect to a, whose zero eigenstate is the radiation ground state occurring in nature, the properties of $|\alpha\rangle_g$ are quite different from $|\alpha\rangle$ in many respects, as presented in this section. Since the term coherent state is usually understood in the original sense (2.1), a new terminology is required for finer distinction between coherent states in the broad sense. The term two-photon coherent states[36] is adopted because of the close connection between these states and quadratic Hamiltonians or two-photon lasing processes, discussed below.

IV. QUADRATIC HAMILTONIAN AND STRUCTURE OF U_L

From (3.11) and (3.20) we can see that $U_L^{(n)}(\alpha^*, \alpha)$ is an exponentiation of a quadratic form in (α^*, α). Such a $U_L^{(n)}(\alpha^*, \alpha)$ can be written as $\langle \alpha | e^{iM} | \alpha\rangle$ for a self-adjoint operator M quadratic in $(a^\dagger, a)$, from a theorem of McCoy.[37] Therefore U_L must be the unitary operator generated by a quadratic Hamiltonian H_Q. The mathematical solution $U_Q(t, t_0)$ of the Schrödinger equation

$$i\hbar \frac{\partial U_Q}{\partial t} = H_Q U_Q, \quad U_Q(t_0, t_0) = 1, \qquad (4.1)$$

is investigated in some detail in this section, which gives all of the U_L.

A. Positive definite quadratic Hamiltonian

Consider the general quadratic Hamiltonian

$$H_Q = \hbar(f_1 a^\dagger a + f_2^* a^2 + f_2 a^{\dagger 2} + f_3^* a + f_3 a^\dagger), \qquad (4.2)$$

where the c numbers f_i may be time dependent. From Hermiticity f_1 has to be real. For a mode of frequency ω, we have $f_1 = \omega$, and $\hbar f_1 a^\dagger a$ represents the free radiation energy of that mode. The f_2 and f_3 terms then represent interaction energies. In particular, the f_2 terms describe a two-photon mechanism, whereas the f_3 terms describe the usual one-photon or linear driving mechanism.

If a physical system with Hamiltonian given by (4.2) is stable, H_Q should not have a spectrum ranging through the entire continuum from $-\infty$ to ∞ in order for a ground state to be defined, i.e., the spectrum of H_Q should be lower bounded. By a shift of scale one may require that the Hamiltonian be positive definite.[38] If this condition is not satisfied, (4.2) cannot be properly used to de-

scribe a physical system of interest in our present context. It has to be extended to include a more complete description of other relevant factors of importance.

Under the condition

$$f_1 > 2|f_2| \tag{4.3}$$

the Hamiltonian (4.2) can be changed, similar to the Bogoliubov transformation,[39] to the following form:

$$H_Q = f_0 b^\dagger b + f_0(|\mu f_3^* - \nu f_3|^2 + |\nu|^2), \tag{4.4}$$

$$b \equiv \mu a + \nu a^\dagger + \mu f_3^* - \nu f_3, \tag{4.5}$$

$$\mu \equiv [2/f_0(f_1 - f_0)]^{1/2} f^* e^{i\phi}, \tag{4.6a}$$

$$\nu \equiv [(f_1 - f_0)/f_0]^{1/2} e^{i\phi}, \tag{4.6b}$$

$$f_0 \equiv (f_1^2 - 4|f_2|^2)^{1/2}, \tag{4.7}$$

where ϕ is an arbitrary phase. The canonical transformation (4.5) is an inhomogeneous linear one, with (4.6) obeying (3.2). The Hamiltonian (4.4) is manifestly positive definite, and (4.3) is indeed the necessary and sufficient condition for a classical H_Q of the form (4.2) to be positive definite.

Similar to the situation of an interacting Bose fluid, $b^\dagger$ of (4.5) can be interpreted as the quasiboson creation operator with a quasiparticle spectrum f_0. The quasiparticle ground state is the $|0_g\rangle$ of (3.7), whereas $|m_g\rangle$ are the excited states. It seems highly unlikely, however, that the radiation states $|m_g\rangle$ may be generated optically.

The following observation should be made: A linear canonical transformation from a to b can always produce f_2 terms in b from a free Hamiltonian $hf_1 a^\dagger a$. In fact, it is easily checked that an f_2 term results if and only if the ν term mixing $b^\dagger$ is present in the transformation. The transformed Hamiltonian contains a constant term $|\nu|^2$, so that the eigenstates of b have this added excitation energy compared to the true ground state $|0\rangle$. Such a Hamiltonian is clearly not physically equivalent to (4.2). In this sense the operator a is more fundamental than b and further justifies the need for viewing $|\alpha\rangle_g$ as a two-photon coherent state with respect to a.

Our following results on $U_Q(t,t_0)$ are actually valid independent of the condition (4.3). Nevertheless, one should note the significance of the positive definiteness of (4.2).

B. Explicit solution for $U_Q(t,t_0)$

The solution $U_Q(t,t_0)$ can be obtained by normal-ordering techniques,[3] among many possible methods. With (4.2), the solution $U_Q^{(n)}(\alpha^*,\alpha;t,t_0)$ of (2.16) is of the form

$$U_Q^{(n)}(\alpha^*,\alpha;t,t_0) = \exp[A(t) + B(t)\alpha^2 + C(t)\alpha^{*2} + D(t)|\alpha|^2 + E(t)\alpha + F(t)\alpha^*], \tag{4.8}$$

where

$$\frac{dA}{dt} = -2(2f_2^* C + f_2^* F^2 + f_3^* F), \tag{4.9a}$$

$$\frac{dB}{dt} = -if_2^*(D+1)^2, \tag{4.9b}$$

$$\frac{dC}{dt} = -2(4f_2^* C^2 + 2f_1 C + f_2), \tag{4.9c}$$

$$\frac{dD}{dt} = -i(4f_2^* C + f_1)(D+1), \tag{4.9d}$$

$$\frac{dE}{dt} = -i(2f_2^* F + f_3^*)(D+1), \tag{4.9e}$$

$$\frac{dF}{dt} = -i[(4f_2^* C + f_1)F + 2f_3^* C + f_3], \tag{4.9f}$$

with the initial condition

$$A(t_0) = B(t_0) = C(t_0) = D(t_0) = E(t_0) = F(t_0) = 0. \tag{4.10}$$

From (4.8) and (2.14) we get

$$\langle\alpha|U_Q(t,t_0)|\beta\rangle = \exp\{-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2 + A(t) + B(t)\beta^2 + C(t)\alpha^{*2} + [D(t)+1]\alpha^*\beta + E(t)\beta + F(t)\alpha^*\}. \tag{4.11}$$

Equation (4.11) can be written in the form of $\langle\alpha|\beta + \zeta(t); \mu(t)\nu(t)\rangle$, apart from a phase factor. Using (3.20), we find by comparison that

$$U_Q(t,t_0)|\beta\rangle = |\beta + \zeta(t); \mu(t), \nu(t)\rangle e^{i\phi(t)}, \tag{4.12}$$

$$\phi(t) = \Theta(t) + \tfrac{1}{2} i[\beta^*\zeta(t) - \beta\zeta^*(t)], \tag{4.13}$$

for a real-valued (β,ζ) independent function $\Theta(t)$ and

$$A(t) = -\tfrac{1}{2}\ln\mu(t) + \nu^*(t)\zeta^2(t)/2\mu(t) - \tfrac{1}{2}|\zeta(t)|^2 + i\Theta(t), \tag{4.14a}$$

$$B(t) = \nu^*(t)/2\mu(t), \tag{4.14b}$$

$$C(t) = -\nu(t)/2\mu(t), \quad D(t) = 1/\mu(t) - 1, \tag{4.14c}$$

$$E(t) = \nu^*(t)\mu(t)/\zeta(t) - \zeta^*(t), \quad F(t) = \zeta(t)/\mu(t). \tag{4.14d}$$

If (4.14) is substituted into (4.9), the equations of motion for μ, ν, ζ, and Θ are obtained,

$$\frac{d\mu}{dt} = if_1\mu - 2if_2^*\nu, \quad \frac{d\nu}{dt} = -if_1\nu + 2if_2\mu, \tag{4.15}$$

$$\frac{d\zeta}{dt} = i(f_3^*\nu - f_3\mu),$$

$$\frac{d\Theta}{dt}=\tfrac{1}{2}f_1+\frac{i}{2}\left[\zeta\left(\frac{d\zeta^*}{dt}\right)-\zeta^*\left(\frac{d\zeta}{dt}\right)\right]. \tag{4.17}$$

Equations (4.15)–(4.17) are equivalent to (4.9). The initial condition (4.10) becomes

$$\mu(0)=1,\quad \nu(0)=0, \tag{4.18}$$

$$\zeta(0)=0, \tag{4.19}$$

while

$$\Theta(0)=0 \tag{4.20}$$

is obtained from (4.12). The general solution (4.8) or (4.12) can be given in quadrature once the solution for (4.15) is given in quadrature, since the other equations can then be immediately integrated.

Defining

$$\mathfrak{F}(t,t_0)\equiv\int_{t_0}^{t}f_1(\tau)\,d\tau, \tag{4.21}$$

$$f\equiv 2if_2e^{2i\mathfrak{F}(t,t_0)}, \tag{4.22}$$

the solution of the first-order coupled differential equation (4.15) under (4.18) is equivalent to the solution of the following equation for a new variable $\mathfrak{C}$:

$$\frac{d\mathfrak{C}}{dt}=f^*\mathfrak{C}^2-f,\quad \mathfrak{C}(0)=0, \tag{4.23}$$

$$\mathfrak{C}\equiv 2C\,e^{2i\mathfrak{F}(t,t_0)}. \tag{4.24}$$

Once C is obtained through (4.23) and (4.24), all of the other variables A, B, D, E, and F can be given in quadratures from (4.19). But Eq. (4.23), a case of the well-studied Riccati equation, does not admit a solution in quadrature in general, although many methods and results on its solution are available.[40,41]

The existence and uniqueness of a solution to (4.23) imply that the solution of (4.9) and (4.10) also exists and is unique. Thus (4.8) is indeed the solution we seek. With $U^{(n)}(\alpha^*,\alpha)$ given by (4.8), the explicit operator expression of $U_Q(t,t_0)$ is

$$U(t,t_0)=e^{A(t)}\,e^{C(t)a^{\dagger2}+F(t)a^\dagger}\,e^{\ln[D(t)+1]a^\dagger a}\,e^{B(t)a^2+E(t)a}\,. \tag{4.25}$$

Equation (4.25) can also be written as

$$\exp(\gamma_1a^{\dagger2}+\gamma_2a^\dagger a+\gamma_3a^2+\gamma_4a^\dagger+\gamma_5a)$$

for a set of c numbers γ_i.[37]

C. Examples

Equation (4.23) can be integrated when f/f^* is independent of t, i.e., when

$$f_2(t)=r(t)\exp[i\phi_2-2i\mathfrak{F}(t,t_0)], \tag{4.26}$$

ϕ_2 independent of t, for a real-valued function $r(t)$ which can take on both positive and negative values. Under (4.26), one finds readily from (4.23), (4.24), (4.9d), (4.14c), and (4.18) that

$$\mu(t)=e^{i\mathfrak{F}(t,t_0)}\cosh[2\mathfrak{R}(t)], \tag{4.27a}$$

$$\nu(t)=ie^{i\phi_2-i\mathfrak{F}(t,t_0)}\sinh[2\mathfrak{R}(t)], \tag{4.27b}$$

$$\mathfrak{R}(t)\equiv\int_{t_0}^{t}r(\tau)\,d\tau, \tag{4.28}$$

$$\zeta(t)=i\int_{t_0}^{t}(f_3^*\nu-f_3\mu)\,dt. \tag{4.29}$$

Equation (4.26) is especially interesting because of the resonant factor $e^{-2i\omega(t-t_0)}$ obtained when $f_1=\omega$. This factor is automatically present in our later physical interpretation of (4.2). The noise behavior in this case is of great interest. Thus when (with also $f_1=\omega$, $t_0=0$)

$$\phi_2=\tfrac{1}{2}\pi, \tag{4.30}$$

$r=$const independent of t, we have from (3.61) and (4.27)

$$\langle\Delta a_1'^2(t)\rangle=\tfrac{1}{4}e^{-4rt},\quad \langle\Delta a_2'^2(t)\rangle=\tfrac{1}{4}e^{4rt}, \tag{4.31}$$

for $a'=ae^{i\omega t}$. In this case the noise in one quadrature component, apart from the optical oscillation, decreases exponentially while the conjugate component has exponentially increasing noise. The quadrature components a_1 and a_2 are completely symmetrical here. We can choose the sign of r, or equivalently the phase of r, to cause either one of them to have decreasing noise. For $\phi_2=0$, $r=$const, we again have (4.31) with $a'=ae^{i(\omega t-\pi/4)}$. The nature of phase control in a' from the phase of f_2 should be clear in this example.

Equation (4.15) can be integrated for a constant f_2 independent of time, with

$$\begin{aligned}\mu(t)&=\cos(\lambda t)+(i\omega/\lambda)\sin(\lambda t),\\ \nu(t)&=i(2f_2/\lambda)\sin(\lambda t),\end{aligned} \tag{4.32}$$

for $\lambda\equiv(\omega^2-4|f_2|^2)^{1/2}\geq 0$. The quantum noise is purely oscillatory in this situation, without attenuation in time.

Equation (4.15) or (4.23) may be integrated in many other situations. A way of generating such example is discussed below.

D. Properties of $U_Q(t,t_0)$

A number of general features exhibited by $U_Q(t,t_0)$ are now observed. From (4.15) we have

$$\mu^*\frac{d\mu}{dt}=\nu\frac{d\nu^*}{dt}+if_1, \tag{4.33}$$

which is a complex equation. The real part of this equation is implied by (3.2), but the imaginary part

is an additional constraint on the pair $\mu(t)$, $\nu(t)$ obtainable from an H_Q. Thus not every pair of functions $\mu(t)$, $\nu(t)$ obeying (3.2) can occur in a $U_Q(t,t_0)$. Only those obeying (4.33) are permitted in the form (4.12). However, this does not impose any constraint on the mean-square fluctuation behavior obtainable in (3.61), since only $|\mu|$, $|\nu|$ are involved here. Moreover, (4.27) shows that *any* particular pair of values (μ,ν) obeying (3.2) can be obtained at a specific t with a proper choice of f_2, even under (4.3) with f_1 fixed.

Any desirable pair of functions $\mu(t)$, $\nu(t)$ obeying (4.33) and (3.2) can be generated by an f_2 through (4.15). This corresponds to the solution of the Riccati equation (4.23) for various f. For example, we can assume ($f_1=\omega$, $t_0=0$)

$$\mu(t)=e^{i\phi(t)}\cosh(\gamma t),\quad \nu(t)=e^{i\phi(t)}\sinh(\gamma t), \tag{4.34}$$

and determine the phase $\phi(t)$ so that (4.33) is satisfied. Equation (4.34) has the interesting characteristic that

$$\langle\Delta a_1^2(t)\rangle=\tfrac{1}{4}e^{-2\gamma t},\quad \langle\Delta a_2^2(t)\rangle=\tfrac{1}{4}e^{2\gamma t}, \tag{4.35}$$

without any shift of phase to a'. Substitution of (4.34) into (4.33) shows that $\phi(t)$ obeys the differential equation

$$\frac{d\phi}{dt}=\frac{1}{\cosh^2(\gamma t)+\sinh^2(\gamma t)},$$

so that

$$\phi(t)=(\omega/\gamma)\tan^{-1}[\tanh(\gamma t)]. \tag{4.36}$$

The corresponding $f_2(t)$ is given by

$$f_2=\tfrac{1}{2}\tanh(2\gamma t)-i\gamma/2\omega. \tag{4.37}$$

Any arbitrary state $|\psi_0\rangle$ will evolve into

$$|\psi_t\rangle=U_Q(t,t_0)|\psi_0\rangle=\int U_Q(t,t_0)|\beta\rangle\langle\beta|\psi_0\rangle d^2\beta/\pi \tag{4.38}$$

under U_Q. If $|\psi_0\rangle=|\beta_0;\mu_0,\nu_0\rangle$, (4.38) can be readily determined from (4.12),

$$U_Q(t,t_0)|\beta_0;\mu_0,\nu_0\rangle =|\beta_0+\zeta(t,t_0);\mu(t,t_0),\nu(t,t_0)\rangle e^{i\phi(t,t_0)}, \tag{4.39}$$

$$\phi(t,t_0)\equiv\Theta(t)+\tfrac{1}{2}i[\beta^*\zeta(t,t_0)-\beta\zeta^*(t,t_0)], \tag{4.40}$$

where

$$\begin{aligned}\mu(t,t_0)&=\mu_0\mu(t)+\nu_0\nu^*(t),\\ \nu(t,t_0)&=\mu_0\nu(t)+\nu_0\mu^*(t),\end{aligned} \tag{4.41}$$

$$\delta(t,t_0)=\mu_0\zeta(t)+\nu_0\zeta^*(t), \tag{4.42}$$

and μ, ν, ζ, and Θ in (4.40)–(4.42) are the solutions to (4.15)–(4.20).

Therefore, under an *arbitrary* $U_Q(t,t_0)$ a TCS will *remain* a TCS for all time, but with β, μ, and ν time varying. By solving $\mu(t)$, $\nu(t)$, and $\zeta(t)$ in terms of μ_0 and ν_0 and $\mu(t,t_0)$, $\nu(t,t_0)$, and $\delta(t,t_0)$, (4.39)–(4.42) and (4.27) show explicitly that *any* $|\beta;\mu,\nu\rangle$ can be obtained from any given $|\beta_0;\mu_0,\nu_0\rangle$ by a $U_Q(t,t_0)$.

It also follows from (4.41) that if f_2 is turned off at time t_f then

$$\mu(t)=\mu(t_f)e^{i\mathfrak{s}(t,t_f)},\quad \nu(t)=\nu(t_f)e^{i\mathfrak{s}(t,t_f)}. \tag{4.43}$$

Thus apart from the optical oscillation the values μ and ν are invariant in the absence of two-photon interaction in H_Q. Furthermore, such invariance is obtained when and only when

$$f_2=0, \tag{4.44}$$

i.e., $\nu(t)=0$. In general, $\mu(t)$ and $\nu(t)$ are determined only by $f_2(t)$ independently of $f_3(t)$, as seen from (4.15). This is in accordance with the fact that the new quantum noise behavior is entirely a consequence of $f_2\neq 0$. Such correlation between f_2 and ν is also observed in Sec. IV A.

On the other hand, the added value $\zeta(t)$ depends on both f_2 and f_3 but is always zero for $f_3=0$,

$$U_Q(t,t_0)|\beta\rangle=|\beta\rangle_g,\quad \text{when } f_3=0. \tag{4.45}$$

Therefore *all* of the U_L's are generated by H_Q with $f_3=0$. The addition of f_3 merely introduces a constant shift to b, as in (4.5). The states (4.39) are eigenstates of

$$b(t)\equiv\mu(t)a(t_0)+\nu(t)a^\dagger(t_0), \tag{4.46}$$

where $a(t_0)$ is the Schrödinger-picture fixed photon annihilation operator. We have

$$b(t)=U_Q(t,t_0)a(t_0)U_Q^\dagger(t,t_0)+\zeta(t)=U_L a U_L^\dagger. \tag{4.47}$$

Equation (4.47) also shows

$$\mu(t)a_H(t)+\nu(t)a_H^\dagger(t)=a(t_0)+\zeta(t) \tag{4.48}$$

for the Heisenberg operator

$$a_H(t)\equiv U_Q^\dagger(t,t_0)a(t_0)U_Q(t,t_0).$$

Equation (4.48) implies, by solving for $a_H(t)$,

$$a_H(t)=\mu^*(t)a(t_0)-\nu(t)a^\dagger(t_0)-\zeta'(t), \tag{4.49}$$

$$\zeta'(t)\equiv\nu(t)\zeta^*(t)-\mu^*(t)\zeta(t). \tag{4.50}$$

Equations (4.49) and (4.39) give

$$U_Q^\dagger(t,t_0)|\beta_0;\mu_0,\nu_0\rangle=|\beta_0+\zeta_r(t);\mu_r(t),\nu_r(t)\rangle e^{i\phi_r(t)}, \tag{4.51}$$

$$\phi_r(t)\equiv-\Theta(t)+\tfrac{1}{2}i[\beta^*\zeta_r(t)-\beta\zeta_r^*(t)], \tag{4.52}$$

$$\mu_r(t)=\mu_0\mu^*(t)-\nu_0\nu^*(t),\quad \nu_r(t)=\nu_0\mu(t)-\mu_0\nu(t), \tag{4.53}$$

$$\zeta_r(t)=\mu_0\zeta'(t)+\nu_0\zeta'^*(t), \tag{4.54}$$

where μ, ν, ζ, and Θ are still the solution to (4.15)–(4.20). The state $U_Q^\dagger(t,t_0)|\beta\rangle_g$ is obtained from evolution of the system backwards in time under the *same* $f_i(t)$ in H_Q, and the operator $b(t)$ is similarly the $a_H(t)$ moving backwards, apart from a constant.

Even though no solution of $U_Q(t,t_0)$ in quadrature can be obtained in general, its structure and behavior have already been fully determined. These behaviors are important in the applications of TCS's.

V. STATE OF TWO-PHOTON LASING RADIATION

We suggest that the output radiation of an ideal monochromatic two-photon laser is in a TCS. In contrast, the output of an ideal one-photon laser is in an ordinary coherent state. Thus TCS's are the characteristic states of two-photon lasing processes (two photons from the *same* mode).

A. Two-photon laser

Two photons of the same frequency ω from the same radiation mode can be absorbed in a single atomic transition between two levels 2 and 1 via an intermediate state, i.e., a transition second order in $\vec{p}\cdot\vec{A}$. Under the usual dipole approximation the parities of levels 2 and 1 have to be identical. First-order dipole transitions between the two levels are forbidden in this circumstance. On the other hand, higher-order multipole contributions to second-order $\vec{p}\cdot\vec{A}$ transitions, which may connect states of different parity, can be significant.[42] First-order transitions between states of opposite parity are also possible via the A^2 term in the radiation interaction Hamiltonian.[42–44] This is possible because in the multipole expansion of the vector potential $A(\vec{r})$, higher-order multipole terms introduce atomic coordinates into the A^2 Hamiltonian. Two-photon absorption spectroscopy is an important experimental technique by now,[45] and many discussions of two-photon absorption processes can be found in the literature.

The radiation-matter (to be referred to as atoms) interaction responsible for the two-photon transition can be expressed in the form

$$H_I = \hbar(pMa^{\dagger 2} + p^* M^\dagger a^2), \tag{5.1}$$

where p is a coupling coefficient and M is the atomic polarization operator which flips the state of the atom If the A^2 mechanism is under consideration, H_I is a true Hamiltonian.[42–44] If virtual transition through an intermediate state has been suppressed through a nonlinear susceptibility, or equivalently through the coupling coefficient p, (5.1) is only an effective Hamiltonian. In this case it can be derived by following a procedure similar to the standard quantum theory of parametric interaction.[46,47] By methods widely used in nonlinear optics,[7] classical equations of motion for the field and the active atomic variables can be obtained in which the virtual intermediate states are accounted for through p. Examples of such treatment for two-photon systems can be found in many places.[48] These classical equations of motion with a quantum-mechanically computed susceptibility can be directly quantized. The effective quantum Hamiltonian follows from the resulting quantum equations of motion.

In addition to absorption, two-photon emission which provides a two-photon lasing mechanism is also described by (5.1). In considering the conditions for stimulated two-photon emission in either the standing-wave or traveling-wave configuration with loss included, it is easily found that stimulated emission cannot be set up from spontaneous emission alone. A sufficiently strong field at the lasing frequency ω has to be present initially.[49–51] Therefore we consider the two-photon lasing configurations depicted in Fig. 2. It is not the purpose of this article to present the semiclassical theory of two-photon laser operation, or to discuss the operating consideration in a realistic material system. But the following facts may be observed:

In the traveling-wave configuration, Fig. 2(a), the gain is obtained from a large external pump field which may have many frequencies but which is still in a single temporal mode. Neglecting various loss mechanisms, this lasing process can be described by the interaction Hamiltonian (5.1) with an initial radiation state ρ_0 having a nonzero amount of power, $\mathrm{tr}\rho_0 a^\dagger a \neq 0$. If a laser is used for the external pump, $\rho_0 = |\alpha\rangle\langle\alpha|$, but intense thermal light can also be used. The radiation state ρ, as well as other dynamical variables, are spatially dependent in this case. In the standing-wave case, Fig. 2(b), only a single standing-wave mode is under consideration. The gain is initially obtained from the usual one-photon lasing mechanism, the host atoms for both lasing transitions being contained in the same cavity.[49] The Hamiltonian is now

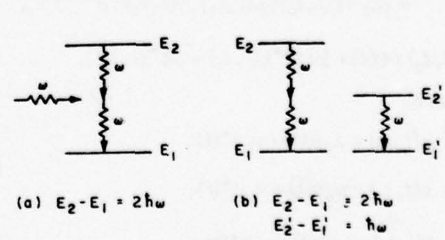

FIG. 2. Two possible configurations for stimulated two-photon emission.

$$H = \hbar(\omega a^\dagger a + p_2 M_2 a^{\dagger 2} + p_2^* M_2^\dagger a^2 + p_1 M_1 a^\dagger + p_1^* M_1^\dagger a), \tag{5.2}$$

with a vacuum initial state; the subscripts 2 and 1 for p and M refer to the two-photon and one-photon lasing material. Population inversion is required in both configurations. Other possible configurations for two-photon lasers can be developed, but these two seem to be the most important.

Spontaneous two-photon emission has been experimentally observed,[52] but not stimulated two-photon emission. Two-photon stimulated emission was first suggested for giant pulse generation.[49]

When such a two-photon laser operates far above threshold with amplitude and phase stability, the fluctuation in M should be relatively small, so that M is approximately a c number. By ignoring the quantum nature of M, (5.2) or (5.3) becomes an H_Q of the form (4.2). If the dissipative mechanisms are also neglected, (4.12) shows that a TSC $|\beta\rangle_g$ will be obtained. A similar argument was used by Glauber[18] to suggest that a coherent state describes one-photon laser radiation far above threshold. As we show in the Appendix, this can be formalized in the so-called self-consistent-field approximation (SCFA), which has been used in one-photon laser theory.[53,54] Therefore a two-photon device of this type operating well above threshold can be expected to produce a state $|\beta\rangle_g$ with further small classically random fluctuations in the variable β, in a way exactly analogous to one-photon lasers where $|\alpha\rangle$ is produced with small classical fluctuations in α. This argument can be made for a standing-wave, a traveling-wave, or an arbitrary spatial-temporal mode.

B. Effect of perturbations in the two-photon system

An additive Gaussian noise and a damping mechanism can be introduced to account for some unavoidable sources of noise similar to the one-photon case, while still preserving that distinguishing feature of TCS's with $\langle\Delta a_1'^2\rangle$ and $\langle\Delta a_2'^2\rangle$ smaller than $\frac{1}{4}$. This can be illustrated as follows: The quantum effect of loss is incorporated through the density operator relaxation equation

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[H_Q, \rho] + \tfrac{1}{2}\gamma(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) + \gamma\bar{n}(a^\dagger\rho a + a\rho a^\dagger - a^\dagger a\rho - \rho a a^\dagger), \tag{5.3}$$

with H_Q given by (4.2). The damping rate is γ and $\bar{n}$ represents the thermal noise contribution from the reservoir that couples to the radiation field. This quantum description of loss has been extensively discussed in the literature.[55–57] For an initial coherent state, (5.3) can be solved exactly through the Fokker-Planck equations for $\rho^{(n)}(\alpha^*, \alpha)$. In particular, all of the characteristic functions are still in Gaussian form,[58] and for $\gamma \neq 4r$,

$$\langle\Delta a_{1,2}'^2(t)\rangle = \frac{1}{4}\frac{\gamma}{\gamma \pm 4r} \pm \frac{r}{\gamma \pm 4r}e^{-(\gamma \pm 4r)t} + \frac{\gamma\bar{n}}{2(\gamma \pm 4r)}(1 - e^{-(\gamma \pm 4r)t}), \tag{5.4}$$

with $a' = ae^{i\omega t}$. It can be readily shown that $\langle\Delta a_j'^2(t)\rangle > 0$ and that the uncertainty principle is obeyed for all t. Note that even when $\bar{n} = 0$, the state is no longer pure in the presence of γ. Asymptotically we have

$$\langle\Delta a_1'^2(t)\rangle \to \gamma\bar{n}/8r, \quad \langle\Delta a_2'^2(t)\rangle \to \infty; \quad t \to \infty, \quad 2r \gg \gamma. \tag{5.5}$$

The condition $2r \gg \gamma$ can be satisfied together with the stability condition (4.3) for the usual values of γ,

$$\omega > 2r \gg \gamma. \tag{5.6}$$

Thus for $\bar{n} \leq 1$ the corresponding state approaches an eigenstate of a_1. In any case the quantum fluctuation in a_1 vanishes as $t \to \infty$.

From the results of Sec. IV B, it can be seen that two-photon laser is a phase-sensitive device. If a uniform random phase is assumed, i.e., if $a' = ae^{j\phi}$ for ϕ uniformly distributed in $[-\pi, \pi]$ is the dynamical observable one has to deal with, then

$$\langle\Delta a_1'^2\rangle = \langle\Delta a_2'^2\rangle = \tfrac{1}{2}(\langle\Delta a_1^2\rangle + \langle\Delta a_2^2\rangle). \tag{5.7}$$

Equation (5.7) contributes a noise greater than $\frac{1}{4}$ for both quadrature components and one distinguishing characteristic of a TCS is lost. The detailed theories of one-photon lasers show that amplitude fluctuation is suppressed in a laser operating far above threshold, and the phase fluctuation is also inversely proportional to the average photon numbers. Moreover, these amplitude stabilization and phase stabilization properties are general features of a wide class of self-sustained oscillators, quantum or classical.[59,55] If the two-photon laser belongs to this class, the phase would diffuse away from some initially measured value only slowly for high-power operation similar to the one-photon case. The phase can then be tracked and the relative phase determined accurately. As a result the field may be regarded to be in a true TCS. Whether the perturbation introduced by the atomic fluctuations may indeed be sufficiently small so that $|\beta\rangle_g$ does not become $|\beta\rangle$ can be determined only by explicitly working out the detailed quantum theory of two-photon lasers. The quantum theory of one-photon lasers has been extensively developed,[55–57] but the two-photon laser requires more careful approximations because of the small

noise in one quadrature component and the more delicate equations of motion involved. Hopefully, the TCS picture can be checked experimentally, as discussed in Sec. VI, by practically realizing a two-photon laser. In any case, $|\beta\rangle_g$ serves as the basic description to be averaged further by a classical density function in the presence of other random interference.

Recently, the quantum theory of multiphoton lasers, including a quantum treatment of the atoms and the loss mechanism, was discussed by McNeil and Walls.[59-61] However, the nature of their approximations was not clarified and they produced lasing results from models[59,60] which cannot lase because of the absence of an input field to provide a nonzero gain. While some qualitative features of their two-photon results are in agreement with the TCS picture, such as large total fluctuation from (3.35), they suggest that the photon statistics are similar to the one-photon case. The Poisson-like distribution they find in Ref. 59 is obtained for an unrealistic loss model, in addition to the above error. The lossless photon distribution they obtain in Ref. 61 is neither Poisson nor of the form (3.23). Since the atomic populations rather than the atomic polarizations were treated as a constant in the derivation of that distribution, it appears that it is valid near, instead of far above, threshold. A complete and careful calculation of the two-photon laser state remains to be carried out.

C. Degenerate parametric amplifier

A degenerate parametric amplifier interpretation can be given to (4.2) similar to the above two-photon laser interpretation. When the material system is suppressed through a susceptibility and when the pump field is treated classically, a two-photon coherent state $|\beta\rangle_g$ would be obtained from a degenerate parametric amplifier such as the one depicted in Fig. 3. An intense initial field at frequency ω has to be present in addition to the pump field for degenerate parametric amplification. A TCS is already present in the treatment of nondegenerate parametric amplifiers by Mollow and Glauber[47] in the joint Wigner distribution for the signal and idler modes, even though a coherent state plus noise is obtained for each of these modes.[62] That a TCS can be generated in a single-mode degenerate parametric amplifier was also observed by Stoler[11] and by Lu.[10]

The phase-sensitive behavior of a two-photon laser is reminiscent of parametric processes, and a degenerate parametric amplifier is a kind of two-photon laser. A basic distinction, however, can be made. In the parametric case the final and initial states of the atoms are identical, but they are different in the two-photon case. The question of quantum statistics is similar in both cases. The pump field of Fig. 3 should be treated as a quantum degree of freedom. A fully quantum treatment of degenerate parametric oscillators can be equally or more complex than a fully quantum treatment of two-photon lasers, depending on whether the atomic variables are included in the parametric case and whether the important intermediate state is included in the two-photon case when present.

No single-mode degenerate parametric oscillation above threshold has been observed. Careful consideration has to be given to any possible competing processes in the actual development of a two-photon laser or degenerate parametric amplifier, such as third-harmonic generation.

VI. COHERENCE AND PHOTON COUNTING

Two-photon coherent states are not coherent according to Glauber's original definition.[63] That is, they do not factorize the correlation functions which describe photon counting experiments employing ordinary photodetectors. However, they factorize a different set of correlation functions obtained by transforming a to b in the field expansion. This kind of transformed correlation function gives the joint counting rates for detectors operating with an interaction mechanism different from the usual dipole absorption,[18] in analogy with detectors that function by stimulated emission discussed by Mandel.[64] Thus a TCS is fully coherent in a sense closely related to the usual definition of optical coherence. Further detailed discussion can be given only in the multimode treatment of TCS.

Here let us consider photon counting experiments associated with a single radiation mode whose spatial dependence has been integrated through the spatial response of a photoabsorption detector. For any state ρ the distribution $\langle n|\rho|n\rangle$ can be generated by an ideal detector if the counting time T is large enough.[65] Therefore we consider the generating function

$$Q(\lambda)=\sum_{n=0}^{\infty}(1-\lambda)^n\langle n|\rho|n\rangle, \tag{6.1}$$

$$\langle a^{\dagger n}a^n\rangle=\langle N\cdots(N-1)\cdots(N-n+1)\rangle \tag{6.2}$$

$$=(-1)^n\frac{\partial^nQ}{\partial\lambda^n}\bigg|_{\lambda=0}. \tag{6.3}$$

With the Glauber model[18] of an n_0-atom photodetector with quantum efficiency s and a sufficiently small counting time T, the generating function for the counting distribution $p(n,T)$ of registering n

counts in time T is merely a scaling of (6.1),

$$p(n,T)=\frac{(-1)^n}{n!}\frac{\partial^n Q(sn_0T\lambda)}{\partial\lambda^n}\bigg|_{\lambda=1}. \tag{6.4}$$

From (3.23) the function $Q(\lambda)$ for a TCS can be evaluated in closed form using the summation formula for Hermite polynomials,[33]

$$Q(\lambda)=\tau^{1/2}\exp\left[[(1-\lambda)\tau-1]\,|\beta|^2 + [1-(1-\lambda)^2\tau]\left(\frac{\nu^*}{2\mu}\beta^2+\frac{\nu}{2\mu^*}\beta^{*2}\right)\right], \tag{6.5a}$$

$$\tau\equiv[\,|\mu|^2-(1-\lambda)^2|\nu|^2]^{-1}. \tag{6.5b}$$

From (6.5) the coefficients $g_n=\langle a^{\dagger n}a^n\rangle/\langle a^\dagger a\rangle^n$ for a first-order coherent field[18] can be computed by (6.3). In particular, $\langle a^\dagger a\rangle=|\nu|^2+|\hat\beta|^2$, as given by (3.29), and

$$\langle a^{\dagger 2}a^2\rangle=\langle a^\dagger a\rangle^2+2|\beta|^2(4|\nu|^2+3)|\nu|^2 - (\mu^*\nu^*\beta^2+\mu\nu\beta^{*2})(1+4|\nu|^2)+|\nu|^2+2|\nu|^4. \tag{6.6}$$

Any value $g_2>0$ can be generated from (6.6) by proper choice of (μ,ν,β). In particular, $g_2<1$ for a TCS if and only if

$$2|\nu|^4+|\nu|^2+2(4|\nu|^2+3)|\nu|^2|\beta|^2 < (\mu^*\nu^*\beta^2+\mu\nu\beta^{*2})(1+4|\nu|^2), \tag{6.7}$$

and $g_2\geq 1$ if the sign is reversed. For $\beta=0$, we always have $g_2>1$ from the noise $|\nu|^2$, but $g_2<1$ is clearly possible for small ν and large μ with nonvanishing β. Recall that $g_2<1$ is a characteristic quantum effect without classical analog, and that currently available sources all generate $g_2\geq 1$. Stoler[11] and Lu[10] have previously observed that $g_2<1$ may be obtained from a state of this kind.

For most values of (μ,ν), g_2 can be tuned by varying β in a TCS from (6.7), exhibiting both anticorrelation ($g_2<1$) and enhanced correlation ($g_2>2$) effects. This fact permits one to check whether a certain particular TCS is obtained by a Hanbury-Brown-Twiss experiment. The photon counting distribution (3.23) or (6.4) is also sufficiently different from a Poisson distribution that its measurement will reveal the presence of a TCS. The influence of nonideal detectors and noise on these photocount statistics should be worked out in detail for actual comparison with experiments.

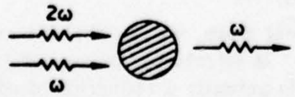

FIG. 3. Phase-matched degenerate parametric amplification with pump field at frequency 2ω.

VII. APPLICATIONS OF TWO-PHOTON COHERENT STATES

We will briefly describe some novel applications of two-photon lasing processes which result from the quantum properties of TCS. Quite apart from their quantum characteristics, two-photon lasers also exhibit certain pulse-shortening behavior absent in the one-photon case.[51] Their particular nonlinearity may lead to many interesting phenomena and applications, including the realization of very-high-power lasers. However, only certain applications of TCS's will be treated here. These applications also illustrate the essential differences between TCS's and ordinary coherent states.

A. Ideal linear amplifier

Under the action of a two-photon amplifier, an input coherent state $|\beta\rangle$ will be changed to $|\beta\rangle_\varepsilon$, from (4.45). For simplicity we assume μ and ν to be real. From (3.28),

$$\langle a_1\rangle_\varepsilon=\hat\beta_1=(\mu-\nu)\beta_1, \tag{7.1a}$$

$$\langle a_2\rangle_\varepsilon=\hat\beta_2=(\mu+\nu)\beta_2. \tag{7.1b}$$

Thus one of the original quadrature components is linearly amplified while the other is linearly attenuated. The nonlinearity of a two-photon amplifier is therefore of the simplest kind; it is piecewise linear.

To be definite let $\nu>0$. The quantum noise $\langle\Delta\beta_1^2\rangle$ decreases from $\langle\Delta\beta_1^2\rangle=\frac{1}{4}$ to $\langle\Delta\hat\beta_1^2\rangle=\frac{1}{4}(\mu-\nu)^2$, while that of $\hat\beta_2$ increases to $\frac{1}{4}(\mu+\nu)^2$, from (3.31). If we define the signal-to-noise ratio

$$(S/N)_{j\rho}=\langle a_j\rangle^2/\langle\Delta a_j^2\rangle,\quad j=1,2, \tag{7.2}$$

where the average is taken in a state ρ, we find that $(S/N)_j$ is invariant in the amplification and attenuation, i.e.,

$$(S/N)_{j|\beta\rangle}=(S/N)_{j|\beta\rangle_\varepsilon}. \tag{7.3}$$

The above definition of signal-to-noise ratio is meaningful. If we try to determine the value of β_2 by measuring a_2 in a coherent state or TCS, (7.2) provides a standard or measure on how well one may estimate β_2,[66,1] or equivalently how "noisy" β_2 is in the "best" measurement.

We can interpret $|\beta\rangle$ as the state of an optical signal presented to a receiver which generates $|\beta\rangle_\varepsilon$ by a two-photon amplifier $U_Q(t,t_0)$ and then measures a_2 on $|\beta\rangle_\varepsilon$. It is obvious physically that the signal-to-noise ratio cannot be improved by amplification. The introduction of f_3 into U_Q does

not increase the signal energy, i.e., the part of which is proportional to β_2^2. The advantage of amplification is practical, similar to the classical case. Raising the signal level makes possible easier measurement of β_2. Since the two-photon amplification process (4.45) operates above threshold, it can be superior to one-photon amplifiers operating below threshold in its possibilities of yielding higher gain and lower noise. If amplification for both quadrature components is desired, the input beam can be first split and two different $U_Q(t,t_0)$ applied to β_1 and β_2 separately. Ideally, no signal-to-noise degradation will occur from (7.3).

The above amplifier performance is not sensitive to the addition of a Gaussian noise. Since we are now utilizing the bigger noise component, there is no need to maintain $\langle \Delta a_1^2 \rangle < \frac{1}{4}$ and a random phase would not seriously degrade the system performance. Therefore the process described by $U_Q(t,t_0)$ provides a possible mechanism in the realization of an ideal linear amplifier.

B. Local oscillator

One may take advantage of the absolutely small $\langle \Delta a_1^2 \rangle \ll \frac{1}{4}$ in $|\beta\rangle_g$ in the reception of a coherent-state signal with a size-limited detector. By employing a local oscillator which generates radiation in state $|\beta\rangle_g$, it is possible to attenuate the quantum noise by an amount which compensates the large diffraction loss incurred in free space or unguided propagation. This is achieved by coupling a mode in $|\beta\rangle_g$ to the incoming mode in $|\alpha\rangle$ in the form of a new mode

$$a_R = \epsilon a_S + (1-\epsilon^2)^{1/2} a_L, \tag{7.4}$$

where ϵ^2 represents the fraction of energy received and a_S, a_L, and a_R are the photon annihilation operators for the signal, the local oscillator, and the receiver mode. The quantum noise in a_R is clearly dominated by that of a_L for small ϵ^2. The signal-to-noise ratio in a_{R1} becomes $\sim\frac{1}{4}\alpha_1^2$ for $\langle \Delta a_{L1}^2 \rangle \simeq 0$, whereas it is $\frac{1}{4}\epsilon^2\alpha_1^2$ for a coherent-state local mode $\langle \Delta a_{L1}^2 \rangle = \frac{1}{4}$. Further discussion can be found in Ref. 1. Note that this advantage of $\langle \Delta a_{L1}^2 \rangle \ll \frac{1}{4}$ cannot be obtained if a one-photon-laser local oscillator is used to produce a coherent-state receiver mode a_R and a $U_Q(t,t_0)$ subsequently applied, because of the corresponding signal attenuation. An absolutely small $\langle \Delta a_{L1}^2 \rangle$ is needed in this scheme.

If an arbitrary amount of power is available, it is possible to have a state with $\langle \Delta a_1^2 \rangle \simeq 0$. However, for a fixed total radiation energy

$$\mathrm{tr}\rho a^\dagger a \le S, \tag{7.5}$$

a decrease of $\langle \Delta a_1^2 \rangle$ in $|\beta\rangle_g$ can be obtained only at the expense of spending a portion of available energy S in the form of added quantum noise energy $|\nu|^2$, from (3.35). Thus $\langle \Delta a_1^2 \rangle \to 0$ requires $S\to\infty$. This consideration is not important in the context of local oscillators at a receiver where large enough amount of power is usually available to make $\langle \Delta a_1^2 \rangle$ sufficiently small. A four-orders-of-magnitude reduction from the quantum noise $\frac{1}{4}$ of α_1 ($|\nu|^2 \simeq 10^2$) for a GHz bandwidth optical signal at $\omega/2\pi \simeq 10^{15}$ Hz requires only a $|\nu|^2$ corresponding to ~1 μW. This will usually bring it down to the level of other extraneous noises. Furthermore, even a ten-orders-of-magnitude reduction of the quantum noise requires ~1 W under the same condition.

Other practical considerations also should not invalidate the utilization of this small-noise property. With proper signaling and reception schemes one can always observe the signal quadrature component without significant degradation from random-phase modulation. From (4.43) it is assured that free-space propagation of $|\beta\rangle_g$ will not disturb the quantum noise characteristics. The rotation (3.59) can be readily compensated in an optical receiver without affecting the signal level, so that TCS's are as good as minimum-uncertainty states in this application. These and many other problems involved in a realistic communication system will be treated elsewhere.

C. Transmission source

Further performance improvement can be obtained if the transmitter generates information-carrying radiation in a TCS. In this case the radiation power constraint (7.5) poses a major limitation on the achievable signal-to-noise ratio. It has been shown that[67] TCS's provide the maximum $(S/N)_1$ obtainable for one of the quadrature components among all possible states satisfying (7.5). By spending a fraction $S/(2S+1)$ of S as quantum noise energy with, e.g.,

$$\mu = (S+1)/(2S+1)^{-1/2}, \quad \nu = S/(2S+1)^{-1/2}, \tag{7.6}$$

$$\beta_2 = \hat{\beta}_2 = 0, \tag{7.7}$$

we have

$$(S/N)_{1\,|\beta\rangle_g} = 4(S^2+S). \tag{7.8}$$

Compared to the signal-to-noise ratio

$$(S/N)_{1\,|\beta\rangle} = 4S \tag{7.9}$$

in a coherent state, we see that the available "signal energy" is effectively increased from S to S^2+S. This is actually a reduction of quantum noise.[68] Among other things, this increase in $(S/N)_1$ leads to a higher information capacity even when the

other quadrature component is not employed for information transmission.[1] In the presence of other extraneous noise, quantum noise reduction in a transmitter state $|\beta\rangle_g$ can also be obtained with only a relatively small $|\nu|^2$, as illustrated above.

Finally, we observe that the new photocount statistics may itself lead to novel applications in communications.

VIII. CONCLUDING REMARKS

We have discussed in some detail the properties of a class of radiation states which are closely related to the well-known coherent states. These two-photon coherent states possess a number of distinguishing characteristics which lead to potential device applications for quantum noise reduction. They may also find application in experimental situations that require low noise sensitivity.

It has also been suggested that these states can be obtained from two-photon stimulated processes, or equivalently that they provide a good description of two-photon laser states. The ultimate validity of this proposition must be tested by an elaborated quantum-statistical theory of two-photon lasing processes and above all, hopefully, by experiment.

Our present work was motivated by the desire to realize physically certain specific quantum states and quantum measurements for practical applications. It appears that the general problems of state and measurement synthesis are interesting and useful areas in quantum optics and electronics, as well as in the foundations of quantum mechanics.

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APPENDIX: TWO-PHOTON LASING STATE IN THE SCFA

Here it is shown that the radiation state of a two-photon laser is a TCS in the self-consistent-field approximation (SCFA),[53,54] if loss is also neglected. Let H be the total Hamiltonian of an atom-field system:

$$H = H_{\text{atom}} + H_{\text{field}} + H_{\text{int}}, \tag{A1}$$

with $H_{\text{field}} + H_{\text{int}}$ given by (5.2). In the SCFA, the total density operator is assumed to factorize,

$$\rho = \rho_{at} \otimes \rho_{fd}. \tag{A2}$$

The equation of motion for the radiation density operator ρ_{fd} can be obtained from that of ρ by tracing over the atomic variables with the result

$$\frac{\partial \rho_{fd}}{\partial t} = -\frac{i}{\hbar}[H_Q, \rho_{fd}], \tag{A3}$$

where H_Q is given by (4.2), with

$$f_1 = \omega, \quad f_2 = p_2 \operatorname{tr}(M_2 \rho_{at}), \quad f_3 = p_1 \operatorname{tr}(M_1 \rho_{at}). \tag{A4}$$

Thus a TCS will be developed from an initial coherent state from (4.12), and (A4) gives the effective field Hamiltonian under the SCFA. Note that a resonant factor $e^{-2i\omega t}$ is automatically present in f_2, from the atomic resonance in M.

Even though the above derivation is given for a single frequency, it applies equally to any spatial-temporal mode if H_I is in the form (5.1). The same result is obtained for a parametric oscillator by interpreting M_1 as the pump photon destruction operator and p_1 as the susceptibility, with $M_2 = 0$. However, it seems less justified to neglect the quantum fluctuation of the pump photon, which is also described by a boson operator as the signal mode.

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Optical Communication with Two-Photon Coherent States - Part I: Quantum State Propagation and Quantum Noise Reduction*

Horace P. Yuen and Jeffrey H. Shapiro**

Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Abstract

Recent theoretical work has shown that novel quantum states, called two-photon coherent states (TCS), have significant potential for improving free-space optical communications. In this three-part study, the communication theory of TCS radiation is explored. The present work, part I, addresses the issues of quantum-field propagation and optimum quantum state generation. In particular, the quantum analog of the classical Huygens-Fresnel-Kirchhoff diffraction theory for quasimonochromatic scalar waves is developed. This result, which describes the propagation of arbitrary quantum states as a boundary-value problem suitable for communication-system analysis, is used to treat a number of quantum transmitter optimization problems. It is shown that under near-field propagation conditions a TCS transmitter maximizes field-measurement signal-to-noise ratio among all transmitter quantum states; the performance

of the TCS system exceeds that for a conventional (coherent state) transmitter by a factor of $N_s + 1$, where N_s is the average number of signal photons (transmitter energy constraint). Under far-field propagation conditions, it is shown that use of a TCS local-oscillator in the receiver can, in principle, attenuate field-measurement quantum noise by a factor equal to the diffraction loss of the channel, if appropriate spatial mode-mixing can be achieved. These communication results are derived assuming that field-quadrature quantum measurement is performed. In part II of this study, photoemissive reception of TCS radiation will be considered; it will be shown therein that homodyne detection of TCS fields can realize the field-quadrature signal-to-noise ratio performance of part I. In part III, the relationships between photoemissive detection and general quantum measurement are established. In particular, a synthesis procedure will be obtained for realizing all the measurements described by arbitrary TCS.

I Introduction

The quantum-mechanical theory of optical communications seeks to delineate the ultimate physical constraints imposed on the transmission of information. The work to date in this field has focused on determining optimum quantum measurements for various detection and estimation problems [1]-[4]. However, to determine the ultimate performance limitations imposed by quantum effects, it is also essential to consider optimum quantum-state generation, because post-measurement statistics (hence detection and estimation performance) depend on both the quantum state and the quantum measurement. Indeed, Yuen has shown that certain "generalized" coherent states of the radiation field possess novel quantum noise characteristics which offer the potential for greatly improved optical communications [5]-[7]. These new states are related to the ordinary coherent states of Glauber, but have not been experimentally observed. They have been called two-photon coherent states, because they can be generated, in principle, by stimulated two-photon processes [7]. In this paper and the sequel [8], [9], we shall consider the use of two-photon coherent state (TCS) radiation in free-space optical communications.

In the usual formulation of quantum communication theory, the electromagnetic field is quantized by representing the positive-frequency field within a receiver region as a sum of

orthogonal space-time modes, and imposing the photon annihilation operator commutation rule on the coefficients in this expansion [4,Ch.5], [10]. This initial-value problem approach is not well-suited to the study of optimum quantum transmitter design. For such a study, it is necessary to have a quantum-mechanical diffraction theory, analogous to the classical Huygens-Fresnel-Kirchhoff theory, in which quantum field propagation in vacuum can be rigorously treated as a boundary-value problem. Because some of the issues of a quantum diffraction theory are not directly related to communication, we will not present a full treatment of such a theory in this paper. On the other hand, we shall develop, in Section II, those features of quantum state propagation which provide the basis for representing any optical channel in a quantum-mechanical form convenient for communication analysis.

The development of quantum diffraction theory will be for arbitrary quantum states; in Sections III-V we shall apply this theory to TCS radiation. Under near-field propagation conditions, with a fixed field-quadrature measurement, it will be shown that a TCS transmitter maximizes post-measurement signal-to-noise ratio over all transmitter quantum states satisfying an average energy constraint. The TCS system substantially outperforms a conventional transmitter of the same energy. Under far-field propagation conditions, with a fixed field-quadrature measurement, it will be shown that use of a TCS local-oscillator in the receiver can, in principle, attenuate post-measurement quantum noise by a factor equal to the diffraction loss of the channel, if appropriate spatial

mode-mixing can be achieved.

In deriving the foregoing communication results, we shall assume that the field-quadrature quantum measurement, i.e., that of a_1 the cosine quadrature of the TCS radiation mode, is performed. When using conventional radiation, the a_1 measurement can be accomplished by homodyning [11]. However, because the statistics of photoemissive detection depend, non-trivially, on the quantum state of the radiation field, the preceding statement regarding homodyning must be re-examined for TCS sources. In [8] we shall present a detailed analysis of photoemissive TCS detection; homodyning will be shown therein to realize the same signal-to-noise ratio as the a_1 measurement for an arbitrary quantum state. The full equivalence of homodyning and the a_1 measurement will be established in [9].

II Quantum Field Propagation

Open almost any graduate text on communication theory [12]-[14], and there, in the introductory chapter, you will find a generic block diagram for a single communication link, much like that shown in Fig. 1. The ubiquity of Fig. 1 stems from its inclusion of virtually all single-link communication problems, e.g., source coding, signal-set selection, channel coding, and optimum receiver principles. Its appearance in the 1970 review article on quantum communication theory by Helstrom *et al*. [1], belies the fact that to that time attention had been focused almost exclusively on receiver optimization. Moreover, Helstrom's recent monograph [4] attests to the continued absence of a quantum-transmitter optimization theory.

A primary cause for the present dearth of quantum-transmitter analysis is, we believe, the cumbersome quantization procedure employed in prior studies. The radiation field is usually quantized at some initial time t_o within a spatial volume V that is large enough to contain the field over the space-time region of interest [10]. In order to use the volume quantization procedure to develop a quantum transmitter theory, we must take V to be a very large spatial volume containing the transmitter, the propagation path, and the receiver. Once the field has been quantized, at an appropriate initial time, within V the quantum characteristics of the receiver entrance-aperture field over the detection interval can in principle be determined.

The foregoing volume-quantization procedure for quantum field propagation [15] is seldom employed. When all the radiation sources produce fields which are in coherent states or their classically-random superpositions, the radiation field in the neighborhood of the receiver can be determined by classical diffraction theory techniques (c.f. Section II.2). On the other hand, for TCS radiation (and other quantum states which do not possess well-behaved P-representations), classical techniques do not suffice. Moreover, to use the volume-quantization procedure in this case we must explicitly describe the radiation source in quantum-mechanical terms. The need to introduce the source dynamical variables into the propagation problem unduly complicates the task of explicitly determining the quantum characteristics of the received field. Thus, our first concern herein will be to formulate a quantum diffraction theory, similar in spirit to its classical counterpart, in order that the quantum field generation and field propagation problems can be separated. Quantum channels can then be conveniently analyzed within the framework of Fig. 1.

II.1 Specification of Quantized Aperture Fields

Classical diffraction theory treats electromagnetic wave propagation as a boundary-value problem. The archetypal free-space diffraction problem is to determine the field over the plane $z=L$, for a wave propagating nominally in the $+z$ direction,[1] given the field over the plane $z=0$. Let $E^{(+)}(\bar{x},z,t)$, where $\bar{r} = (x,y,z)$, $\bar{x} = (x,y)$, be the positive frequency electric field

[1] In other words, the wave under consideration has no components in its angular spectrum which transport power in the $-z$ direction.

operator for a scalar wave[2] that propagates nominally in the +z direction in vacuum. The corresponding quantum diffraction problem is as follows. Given the quantum characteristics of

$$\underset{\sim}{E}_0^{(+)}(\bar{x},t) \equiv \underset{\sim}{E}^{(+)}(\bar{x},0,t) \;, \tag{2.1}$$

determine the quantum characteristics of

$$\underset{\sim}{E}_L^{(+)}(\bar{x},t) \equiv \underset{\sim}{E}^{(+)}(\bar{x},L,t) \;, \tag{2.2}$$

for an arbitrary positive L. Herein we show how to formulate these planar characterizations; the propagation question is addressed in Section II.2.

The classical positive-frequency electric field, $\underset{\sim}{E}^{(+)}(\bar{r},t)$, can be expanded on a fixed transverse plane

$$\Sigma_z = \{(x,y,z): -\infty<x,y<\infty\} \;,$$

and an arbitrary time interval T_z in terms of a complete orthonormal (CON) set of spatio-temporal mode functions $\{\xi_n(\bar{x},t): (\bar{x},t)\epsilon\Sigma_z x T_z\}$, i.e.,

$$\underset{\sim}{E}_z^{(+)}(\bar{x},t) \equiv \underset{\sim}{E}^{(+)}(\bar{x},z,t) = \sum_n e_n \xi_n(\bar{x},t) \;, \tag{2.3}$$

where

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[2] As in classical optics, the scalar wave analysis is not completely rigorous. However, because the generalization of our quantum propagation theory to vector waves involves only added notational burden, we restrict ourselves to the scalar case.

$$\int_{T_z}\int_{\Sigma_z}\xi_n(\bar{x},t)\xi^*_{n'}(\bar{x},t)d\bar{x}dt=\delta_{nn'},$$
$$\sum_n \xi_n(\bar{x},t)\xi^*_n(\bar{x}',t')=\delta(\bar{x}-\bar{x}')\delta(t-t'), \tag{2.4}$$

and $\{e_n\}$ are the classical mode-amplitudes. When this classical field is quantized, the classical mode-amplitudes are changed to a set of quantum operators, $\{\lambda_n a_n\}$, where $\{\lambda_n\}$ is a set of mode-dependent c-numbers, and $\{a_n\}$ is a set of modal photon annihilation operators obeying the following canonical commutation rules (CCR)

$$[a_n, a^\dagger_{n'}] = \delta_{nn'}, \quad [a_n, a_{n'}] = 0. \tag{2.5}$$

The proper way to carry out this quantization procedure is not described in this paper; its relationship to the usual volume quantization procedure will be discussed in a moment.

In general, we choose not to work with the electric field operator obtained by the preceding method, but rather with the field operator

$$E_z(\bar{x},t) = \sum_n a_n\xi_n(\bar{x},t), \tag{2.6}$$

obtained by dropping the λ_n. For quasimonochromatic (narrowband) fields in free space with a nominal radian frequency ω_o, the field (2.6) is proportional to the positive-frequency electric field operator, viz

$$\underset{\sim}{E}_z^{(+)}(\bar{x},t) = (\hbar\omega_o/2\varepsilon_o)^{1/2}E_z(\bar{x},t), \tag{2.7}$$

where ε_o is the vacuum permittivity. The quasimonochromatic condition is not actually required in our theory once it is understood that we are working with the field E_z, rather than the usual electric field $\underset{\sim}{E}_z^{(+)}$. For simplicity, however, quasimonochromaticity will be assumed henceforth.

From (2.4) and (2.6) we have the field commutator

$$[E_z(\bar{x},t)\ ,\ E_z^{\dagger}(\bar{x}',t')] = \delta(\bar{x}-\bar{x}')\delta(t-t')\ , \tag{2.8}$$

where $E_z^{\dagger}(\bar{x},t)$, the adjoint operator of $E_z(\bar{x},t)$, is the negative-frequency counterpart of the field operator $E_z(\bar{x},t)$. Equation (2.8) imposes a condition to be fulfilled by the quantized field at any transverse plane, and forms the foundation of our propagation analysis.Apart from a proportionality constant, Helstrom has shown [4, Ch.5] that (2.8) gives the electric-field commutator in the limit of paraxial propagation. More generally, by excluding evanescent waves and augmenting (2.6) to include waves which propagate nominally in the -z direction, we can obtain the usual volume-quantization commutator [4], [10], [15] for the electric-field operator $\underset{\sim}{E}_z^{(+)}$ from a planar-quantization procedure, and vice versa. We shall, however, purposefully retain the evanescent-wave contribution to (2.6) because of its role in preserving the field commutator (2.8) at all planes transverse to the nominal direction of wave propagation. Indeed, the validity of (2.8) at any z-plane will force us to identify the quantum-noise sources responsible for its preservation, thus making evident to what extent we can control the quantum state at the receiver in subsequent communication analyses (cf. Sections IV, V).

A complete specification of the quantized electromagnetic field requires, in addition to (2.4), (2.6), (2.8), knowledge of the density operator, ρ, describing the states of the modes. In the absence of classical statistical dependence, we have that

$$\rho = \bigotimes_n \rho_n , \tag{2.9}$$

where ρ_n describes the state of the n^{th} mode. The state of a quantized aperture field $E_{A_z}(\bar{x},t)$, the field $E_z(\bar{x},t)$ restricted to an aperture A_z in Σ_z, can be given similarly by a modal expansion with a density operator specifying the state of the modal photon operators. If the planar field $E_z(\bar{x},t)$ is specified first, the state of $E_{A_z}(\bar{x},t)$ can be determined from that of $E_z(\bar{x},t)$ by modal transformations (see Appendix).

To solve the propagation problem we must show how the δ-function commutator, (2.8), is preserved and how the density operator is transformed by free-space diffraction.

II.2 Modal Theory of Propagation

Consider the line-of-sight free space optical communication channel, whose propagation geometry is shown in Fig. 2. A quasimonochromatic quantized radiation source produces a field in the z=0 plane that is propagating nominally in the +z direction and is space-limited to a transmitter exit aperture A_0. After propagation through L meters of free space, the field is collected over a receiver entrance aperture A_L. For convenience, we shall assume that A_0 and A_L are circular apertures of diameters d_0 and d_L respectively whose centers lie on the z-axis. The quantum propagation problem is simply stated: given the quantum field

produced by the source over the transmitter aperture, determine the quantum field that results over the receiver aperture. A more precise formulation follows.

Suppose the source produces a field in the transmitter pupil that is time limited to an interval

$$\mathcal{T}_0 = \{t:\ t_o - T \le t \le t_o\} .$$

We denote the field operator in the transmitter aperture by $E_{A_0}(\bar{x},t)$. This operator has the mode expansion

$$E_{A_0}(\bar{x},t) = \sum_n a_n \zeta_n^0(\bar{x},t) , \tag{2.10}$$

where $\{\zeta_n^0\}$ is a CON set on $A_0 \times \mathcal{T}_0$, and the $\{a_n\}$ satisfy the CCR. To complete the quantum aperture field specification, we assume the $\{a_n\}$ in (2.10) have a known density operator ρ_{A_0}.

In a similar manner, we assume that the receiver time limits the field in its entrance aperture to an interval, $\mathcal{T}_L$, that is sufficient to encompass the entire source transmission. We denote the field operator in the receiver aperture by $E_{A_L}(\bar{x},t)$, which we expand as follows

$$E_{A_L}(\bar{x},t) = \sum_n b_n \zeta_n^L(\bar{x},t) . \tag{2.11}$$

In Eq. (2.11) the $\{\zeta_n^L\}$ form a CON set on $A_L \times \mathcal{T}_L$, and, according to

our planar quantization procedure, the $\{b_n\}$ must satisfy the CCR. To solve the propagation problem we must determine ρ_{A_L}, the density operator for the $\{b_n\}$, from ρ_{A_0}. Our solution to this quantum field problem must subsume the classical field behavior, i.e., the average received field

$$\mathcal{E}_{A_L}(\bar{x},t) \equiv \mathrm{Tr}\{\rho_{A_L} E_{A_L}(\bar{x},t)\} \tag{2.12}$$

must be given in terms of the average transmitted field

$$\mathcal{E}_{A_0}(\bar{x},t) \equiv \mathrm{Tr}\{\rho_{A_0} E_{A_0}(\bar{x},t)\} \tag{2.13}$$

by the classical diffraction formula .

Conceptually, it is convenient to solve this propagation problem by a three-step procedure:

i) determine from $E_{A_0}(\bar{x},t)$ and ρ_{A_0} the quantum characteristics of $E_0(\bar{x},t)$, the field operator over the entire z=0 plane, i.e., for $(\bar{x},t) \in \Sigma_0 \times T_0$;

ii) determine from $E_0(\bar{x},t)$ and its density operator, ρ_0, the quantum characteristics of $E_L(\bar{x},t)$, the field operator over the entire z=L plane, i.e., for $(\bar{x},t) \in \Sigma_L \times T_L$;

iii) determine from $E_L(\bar{x},t)$ and its density operator, ρ_L, the quantum characteristics of $E_{A_L}(\bar{x},t)$.

Within the foregoing schema, tasks i) and iii) are trivial. Specifically, we can decompose $E_0(\bar{x},t)$ into $E_{A_0}(\bar{x},t)$ plus a field operator defined on $(\Sigma_0 - A_0) \times T_0$. All the spatio-temporal modes

of the latter operator must be in the vacuum state because we assume that the source acts on the half space $z>0$ only through the aperture A_0, and we assume that there is no classical interference present, such as thermal noise. Thus if we expand $E_0(\bar{x},t)$ in the form

$$E_0(\bar{x},t)=\sum_n (a_n\zeta_n^0(\bar{x},t)+c_n\zeta_n^c(\bar{x},t)) , \tag{2.14}$$

where $\{\zeta_n^c\}$ are CON on $(\Sigma_0-A_0)\times T_0$, and the $\{c_n\}$ satisfy the CCR, we find that task i) is completed taking

$$\rho_0=\rho_{A_0}\bigotimes_n|0\rangle\langle 0| . \tag{2.15}$$

Task iii) is conceptually even simpler; it corresponds to extracting from a quantum planar field on $\Sigma_L\times T_L$ the spatio-temporal modes that span $A_L\times T_L$. Task ii) will be accomplished by exploiting the classical mode theory of diffraction. The following classical results are well known [16],[17]; they are presented to establish notation .

A. Classical Fields

As in (2.1), (2.2) let us use $\mathcal{E}_0(\bar{x},t)$ and $\mathcal{E}_L(\bar{x},t)$ to denote the classical fields associated with $E_0(\bar{x},t)$ and $E_L(\bar{x},t)$ respectively. In the quasimonochromatic limit in which we are working $\mathcal{E}_0$ and $\mathcal{E}_L$ can be uniquely decomposed into their propagating and evanescent wave components, viz

$$\mathcal{E}_0(\bar{x},t)=\mathcal{E}_0^p(\bar{x},t)+\mathcal{E}_0^e(\bar{x},t) , \tag{2.16}$$

and $$\mathcal{E}_L(\bar{x},t)=\mathcal{E}_L^p(\bar{x},t)+\mathcal{E}_L^e(\bar{x},t) , \tag{2.17}$$

where the two-dimensional spatial Fourier transforms of the propagating-wave ($\mathcal{E}^p$) components vanish for spatial frequencies beyond $\omega_0/2\pi c$ in magnitude and those of the evanescent-wave ($\mathcal{E}^e$) components vanish for spatial frequencies below $\omega_0/2\pi c$ in magnitude.

The propagating-wave component of $\mathcal{E}_L$ is generated by the propagating-wave component of $\mathcal{E}_0$ through the quasimonochromatic diffraction integral

$$\mathcal{E}_L^p(\bar{x},t) = \int_{T_0} dt' \int_{\Sigma_0} d\bar{x}'\ \mathcal{E}_0^p(\bar{x}',t) G_L(\bar{x},t;\bar{x}',t') \ , \qquad (2.18)$$

where the Green's function G_L satisfies

$$G_L(\bar{x},t;\bar{x}',t') = \int d\bar{f}\ \mathrm{circ}\ (\lambda_0 \bar{f})\delta(t-t'-L/c(1-|\lambda_0\bar{f}|^2)^{1/2})$$

$$\cdot \exp\{-j2\pi[L\lambda_0^{-1}(1-|\lambda_0\bar{f}|^2)^{1/2} - \bar{f}\cdot(\bar{x}-\bar{x}')]\} \ . \qquad (2.19)$$

In (2.19), λ_0 is the wavelength associated with ω_0 and $\bar{f}$ is a two-dimensional spatial frequency. The evanescent-wave component of $\mathcal{E}_L$ will be nearly zero, regardless of $\mathcal{E}_0$, for the usual case of $L \gg \lambda_0$.

Inasmuch as $\mathcal{E}_{A_L}$ is the restriction of $\mathcal{E}_L$ to $A_L \times T_L$, and the boundary condition on z=0 is $\mathcal{E}_0(\bar{x},t) \equiv 0$ for $(\bar{x},t) \notin A_0 \times T_0$, the foregoing results imply that

$$\mathcal{E}_{A_L}(\bar{x},t) = \int_{T_0} dt' \int_{A_0} d\bar{x}'\ \mathcal{E}_{A_0}(\bar{x}',t') G_L(\bar{x},t,;\bar{x}',t') \ . \qquad (2.20)$$

Now, because G_L is a compact operator [17], [18] that maps $L^2(A_0 \times T_0)$ into $L^2(A_L \times T_L)$ via (2.20), it possesses a modal decomposition of the form [3]

$$G_L(\bar{x},t;\bar{x}',t') = \sum_n \eta_n^{1/2} \zeta_n^L(\bar{x},t) \zeta_n^{0*}(\bar{x}',t) , \qquad (2.21)$$

where $\{\zeta_n^L\}$ and $\{\zeta_n^0\}$ are CON sets, so-called output and input eigenfunctions, on $A_L \times T_L$ and $A_0 \times T_0$ respectively. These eigenfunctions are determined from the Fredholm equations

$$\int_{T_0} dt_2 \int_{A_0} d\bar{x}_2 \, K_L(\bar{x}_1,t_1;\bar{x}_2,t_2) \zeta_n^0(\bar{x}_2,t_2) = \eta_n \zeta_n^0(\bar{x}_1,t_1) , \qquad (2.22)$$

$$\int_{T_0} dt' \int_{A_0} d\bar{x}' \, \zeta_n^0(\bar{x}',t') G_L(\bar{x},t;\bar{x}',t') = \eta_n^{1/2} \zeta_n^L(\bar{x},t) , \qquad (2.23)$$

where the kernel K_L is

$$K_L(\bar{x}_1,t_1;\bar{x}_2,t_2) = \int_{T_L} dt' \int_{A_L} d\bar{x}' \, G_L^*(\bar{x}',t';\bar{x}_1,t_1) G_L(\bar{x}',t';\bar{x}_2,t_2) . \qquad (2.24)$$

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[3] Strictly speaking, to make G_L a compact operator we should replace the δ-function in (2.19) by the bandlimited δ-function $W \operatorname{sinc}[W(t-t' - L/c(1-|\lambda_0 \bar{f}|^2)^{1/2})]$ for some finite W. The parameter W is the bandwidth of the quasimonochromatic radiation. As the proof of the quantum propagation theorem and all subsequent manipulations can be carried out with some additional labor using the bandlimited G_L, we have chosen to present the simpler formulation.

The eigenvalue, η_n, associated with nth input-output eigenfunction pair is, physically, the fractional energy transfer from $A_0 \times T_0$ to $A_L \times T_L$ that results when we transmit $\zeta_n^0(\bar{x},t)$. Because the free-space channel is a passive non-absorbing propagation medium, we have

$$0 \leq \eta_n \leq 1 \text{ , for all n.} \qquad (2.25)$$

There are two loss mechanisms that make $\eta_n < 1$. First, there is the radiative loss that is incurred when A_L is not large enough to encompass the entire z=L plane field that results from transmission of $\zeta_n^0(\bar{x},t)$. Second, there is the evanescence loss that is incurred because $\zeta_n^0(\bar{x},t)$ contains spatial-frequency components beyond λ_o^{-1} in magnitude.

By use of the mode decomposition for G_L we can replace the diffraction integral (2.20) with an equivalent parallel-channel (modal) description. Specifically, if we expand ε_{A_0} using the input eigenfunctions $\{\zeta_n^0\}$ as the basis set, viz

$$\varepsilon_{A_0}(\bar{x},t) = \sum_n \alpha_n \zeta_n^0(\bar{x},t) \qquad (2.26)$$

we find that $\mathcal{E}_{A_L}$ has the expansion

$$\mathcal{E}_{A_L}(\bar{x},t) = \sum_n \alpha_n \eta_n^{1/2} \zeta_n^L(\bar{x},t) \qquad (2.27)$$

in terms of the output eigenfunctions $\{\zeta_n^L\}$.

B. Quantum Fields

Just as in the classical case, let us decompose the field operators E_0 and E_L into their propagating and evanescent wave components, i.e.

$$E_0(\bar{x},t) = E_0^p(\bar{x},t) + E_0^e(\bar{x},t) \ , \qquad (2.28)$$

$$E_L(\bar{x},t) = E_L^p(\bar{x},t) + E_L^e(\bar{x},t) \ . \qquad (2.29)$$

We are interested in developing the quantum characteristics of E_L given those of E_0. From the classical description of the propagation process, it is apparent that E_L^e is not influenced by E_0 in the usual case of $L \gg \lambda_o$. Thus, the annihilation operators in any mode expansion of E_L^e must all be in the vacuum state, and we can concentrate our attention on E_L^P.

$$E_{A_L}(\bar{x},t)=\sum_n b_n \zeta_n^L(\bar{x},t) \ , \tag{2.32}$$

in terms of the classical output eigenfunctions, then the photon operators $\{b_n\}$ satisfy

$$b_n = a_n \eta_n^{1/2} + c_n(1-\eta_n)^{1/2} \ , \tag{2.33}$$

where $\{\eta_n\}$ are the classical eigenvalues and $\{c_n\}$ are vacuum-state photon operators. The density operator ρ_{A_L} for the $\{b_n\}$ is determined from ρ_{A_0} and (2.33) through (A.2), (A.4). In particular, if

$$\rho_{A_0} = \bigotimes_n \rho_{A_0 n} \tag{2.34}$$

we obtain

$$\rho_{A_L} = \bigotimes_n \rho_{A_L n} \tag{2.35}$$

where for each n $\rho_{A_L n}$ is computed from $\rho_{A_0 n}$ via (A.9).

<u>Proof:</u>

We have, from (2.28)-(2.30) and the surrounding discussion, that

$$E_L(\bar{x},t) = \int_T dt' \int_{\Sigma_0} d\bar{x}' E_0(\bar{x}',t') G_L(\bar{x},t;\bar{x}'t') + E_L^e(\bar{x},t) \ . \tag{2.36}$$

Restricting (2.36) to $A_L \times T_L$ and using the fact that $E_0(\bar{x},t)=E_{A_0}(\bar{x},t)$ on $A_0 \times T_0$ we find

$$E_{A_L}(\bar{x},t)= \int_{T_0} dt' \int_{A_0} d\bar{x}' E_{A_0}(\bar{x}';t) G_L(\bar{x},t;\bar{x}',t')$$

$$+ \iint_{(\Sigma_0 \times T)-(A_0 \times T_0)} dt' d\bar{x}' E_0(\bar{x}',t') G_L(\bar{x},t;\bar{x}',t') + E_L^e(\bar{x},t) . \qquad (2.37)$$

We now use (2.37) to evaluate the series (2.32) and obtain with the aid of (2.21)

$$b_n = a_n \eta_n^{1/2} + d_n \qquad (2.38)$$

where d_n is the operator

$$d_n = \int_{T_L} dt \int_{A_L} d\bar{x} \zeta_n^{L*}(\bar{x},t) \left[\iint_{(\Sigma_0 \times T)-(A_0 \times T_0)} dt' d\bar{x}' E_0(\bar{x}';t) G_L(\bar{x},t;\bar{x}'t') + E_L^e(\bar{x},t) \right] \cdot (2.39)$$

Because $\{b_n\}$ obey the CCR, it follows that

$$d_n = c_n (1-\eta_n)^{1/2} \exp(j\phi_n) \qquad (2.40)$$

in terms of a set of photon operators $\{c_n\}$ and arbitrary phase factors $\{\phi_n\}$. Furthermore, since $E_0(\bar{x}',t')$ is a vacuum-state field on $(\Sigma_0 x T)-(A_0 x T_0)$ as is $E_L{}^e(\bar{x},t)$ on $A_L x T_L$, the $\{c_n\}$ are all vacuum-state operators. For any $\{\phi_n\}$ the state of the modes described by the photon operators $\{\exp(j\phi_n)c_n\}$ is therefore the vacuum $\bigotimes_n |0\rangle\langle 0|$, so (2.33) and the theorem follow immediately with the inconsequential choice $\phi_n=0$; the density operator transformation is given in the Appendix.

Remarks:

The basic physical content of the quantum propagation theorem is as follows. For E_{A_0} and E_{A_L} expanded in the series (2.31), (2.32) with

$$\mathrm{Tr}(\rho_{A_0} a_n) = \alpha_n , \qquad (2.41)$$

the classical mode theory of diffraction implies that

$$\mathrm{Tr}(\rho_{A_L} b_n) = \alpha_n \eta_n^{1/2} \qquad (2.42)$$

Now, if $\eta_n<1$ we cannot have

$$b_n = a_n \eta_n^{1/2} ,$$

because b_n would violate the CCR. There has to be, in this case, a photon operator c_n, that satisfies the CCR, which contributes to b_n according to (2.33). Physically, when $\eta_n<1$ the output eigenfunction $\zeta_n^L(\bar{x},t)$ couples to modes that are not generated by

$E_{A_0}(\bar{x},t)$. Because all such fields are in vacuum state, (2.33) is consistent with the classical mode result (2.42). The development in Section V provides a detailed example of the role such auxiliary modes play in preserving the commutator (2.8), as well as the effect of auxiliary modes on the extent to which the receiver state can be controlled.

The quantum propagation theory developed in this section is actually more general than the preceding theorem indicates. In particular, it can be applied to arbitrary linear propagation media, and can handle the case of non-vacuum c_n in (2.33), c.f. Section V.1.

C. Coherent State Propagation

An ordinary optical source is one whose radiation field is in a Glauber coherent state or a classically random superposition of such states. In free-spece quantum receiver theory, quantization of such a source is generally performed (conceptually) in the receiver entrance aperture or fictitious measurement cavity, even though the field is physically quantum mechanical in the neighborhood of the source. By means of the quantum propagation theorem we can rigorously justify the use of classical propagation theory for ordinary sources.

Suppose that the quantized transmitter field, $E_{A_0}(\bar{x},t)$, has a density operator in the modal expansion (2.13) whose P-representation possesses a classical analog, i.e., it has the coherent state expansion

$$\rho_{A_0} = \int (\bigotimes_n |\alpha_n\rangle\langle\alpha_n|) P(\alpha_1,\alpha_2,\cdots\alpha_n,\cdots;\alpha_1^*,\alpha_2^*,\cdots\alpha_n^*,\cdots)\prod_n d^2\alpha_n/\pi \,, \quad (2.4$$

where $P \prod_n 1/\pi$ is a classical probability density. The quantum propagation theorem (2.31)-(2.33) in conjunction with the density-operator transformation calculus presented in the Appendix can be readily used to show that the receiver field, $E_{A_L}(\bar{x},t)$, has the density operator

$$\rho_{A_L} = \int \left(\bigotimes_n |\eta_n^{1/2}\alpha_n\rangle\langle\eta_n^{1/2}\alpha_n|\right) P(\alpha_1,\alpha_2,\cdots\alpha_n,\cdots;\alpha_1^*,\alpha_2^*,\cdots\alpha_n^*,\cdots)\prod_n d^2\alpha_n/\pi \quad ,(2.44)$$

in the modal expansion (2.32)[4]. Equations (2.43), (2.44) demonstrate that the received field behaves, quantum mechanically, as though the propagation were classical and quantization were applied at the receiver. We shall see, in the sequel, that the quantum propagation theorem has signigicant content for more general quantum transmitters.

[4] Equation (2.44) has to be modified in the presence of classical interference or noise, which is not our concern here.

III Two-Photon Coherent States

We now introduce a class of radiation states that are closely related to ordinary coherent states, but which possess a number of advantages from a communication viewpoint. A comprehensive discussion of the mathematical and physical nature of these states has been given elsewhere[7], so that the present discussion serves primarily to establish notation and indicate the possible communication implications.

III.1 State Specification and Generation

For a single radiation mode of frequency ω_0 with photon annihilation operator a, the two-photon coherent states(TCS) $|\beta\rangle_g$ are defined to be the eigenstates of a transformed photon operator b with eigenvalues β. Specifically, we define

$$b = \mu a + \nu a^\dagger \quad , \tag{3.1}$$

for complex μ and ν such that

$$|\mu|^2 - |\nu|^2 = 1 \; ; \tag{3.2}$$

the TCS $|\beta\rangle_g$ satisfies

$$b|\beta\rangle_g = \beta|\beta\rangle_g \; . \tag{3.3}$$

When $\nu=0$, $|\beta\rangle_g$ becomes the ordinary coherent state $|\beta\rangle$. For non-zero ν, the state $\rho = |\beta\rangle_{gg}\langle\beta|$ does not possess a well-behaved P representation.

The TCS$|\beta\rangle_g$ has the following important fluctuation behavior

$$_g\langle\beta|a|\beta\rangle_g = \mu^*\beta - \nu\beta^* \equiv \tilde{\beta} \ , \tag{3.4}$$

$$_g\langle\beta|a^\dagger a|\beta\rangle_g = |\tilde{\beta}|^2 + |\nu|^2 \ , \tag{3.5}$$

$$\langle\Delta a_1{}^2\rangle = |\mu-\nu|^2/4 \ , \ \langle\Delta a_2{}^2\rangle = |\mu+\nu|^2/4 \ , \tag{3.6}$$

where

$$a \equiv a_1 + ja_2 \tag{3.7}$$

for self-adjoint a_1, a_2 defines the quadrature components of the mode in question and $\Delta a_i \equiv a_i - \langle a_i\rangle$ is the fluctuation in a_i, $i=1,2$. (In these expressions, the angular brackets denote quantum expectation with respect to a given density operator.)

For arbitrary μ, ν, the TCS$|\beta\rangle_g$ achieves the minimum uncertainty product

$$\langle\Delta a_1'^2\rangle\langle\Delta a_2'^2\rangle = 1/16 \ , \tag{3.8}$$

for the quadrature components of the photon operator

$$a' \equiv a \exp\left\{ j \tan^{-1}\left(\frac{j(\mu^*\nu - \nu^*\mu)}{2|\mu||\nu| + \mu^*\nu + \nu^*\mu} \right)\right\} \tag{3.9}$$

with

$$\langle \Delta a_1'^2 \rangle = (|\mu| - |\nu|)^2/4 \ , \ \langle \Delta a_2'^2 \rangle = (|\mu| + |\nu|)^2/4 \ . \quad (3.10)$$

When μ/ν is real, TCS are the usual minimum uncertainty states for the Heisenberg inequality $\langle \Delta a_1^2 \rangle \langle \Delta a_2^2 \rangle \geq 1/16$. In the limit $\mu,\nu \to \infty$ with $\mu/\nu \to 1$ and $\mu/\nu \to -1$, TCS become the eigenstates of a_1 and a_2 respectively, viz

$$\begin{aligned} &\langle \Delta a_1^2 \rangle \to 0 \ , \quad \text{for } \mu/\nu \to 1 \ , \\ &\langle \Delta a_2^2 \rangle \to 0 \ , \quad \text{for } \mu/\nu \to -1 \ . \end{aligned} \qquad (3.11)$$

It follows from (3.6) or (3.10) that the quantum fluctuations in a_1 and a_2 (or a_1' and a_2') can be exchanged, at constant minimum uncertainty product, by varying μ,ν in a TCS. For coherent-state radiation we have that $\langle \Delta a_1^2 \rangle = \langle \Delta a_2^2 \rangle = 1/4$, and random superposition of coherent states always increases the fluctuation levels. Thus, the exchange of quadrature-fluctuation levels is <u>not</u> a feature of conventional sources.

Whereas a coherent-state radiation field is generated by a one-photon laser operated well above threshold, a TCS can be generated, in principle, by a two-photon laser operated well above threshold. The theoretical relationship between TCS radiation and stimulated two-photon processes has been elaborated in [7]. As yet, no TCS radiation has been experimentally observed, although the generation of such light remains a physical possibility.

Naively, the rationale for improving free-space optical communication performance through use of TCS radiation is as follows. At the transmitter, we impress analog or digital information onto the average value of the a_1 quadrature of a single TCS radiation mode. By controlling μ,ν we can reduce the quantum fluctuation on the signal (a_1) quadrature below the level for coherent states. Thus, measurement of a_1 at the receiver yields a higher signal-to-noise ratio for analog communication or lower error probability for digital communication than that obtained with a coherent-state transmitter of the same energy.

III.2 State Propagation

The preceding description of potential TCS communication advantages is oversimplified because it neglects the effects of propagation from the transmitter to the receiver. Consider the line-of-sight free space propagation channel that was analyzed in Section II. Suppose that a TCS transmitter produces a field $E_{A_0}(\bar{x},t)$ whose state, ρ_{A_0}, in the expansion (2.31) is

$$\rho_{A_0} = |\beta\rangle_{gg}\langle\beta| \bigotimes_n |0\rangle\langle 0| \quad , \tag{3.12}$$

i.e., the first input eigenfunction is in a TCS and all the remaining eigenfunctions are in vacuum state. Via the quantum propagation theorem we can separate the state generation and propagation problems and find that $E_{A_L}(\bar{x},t)$, the receiver field, has state

$$\rho_{A_L} = \rho_1 \bigotimes_n |0\rangle\langle 0| \tag{3.13}$$

in the expansion (2.32). In (3.13), ρ_1 is the state of the first output eigenfunction; it is the state of a photon operator

$$b = \eta_1^{1/2} a + (1-\eta_1)^{1/2} c \qquad (3.14)$$

where a, c are photon operators that are in states $|\beta\rangle_g$ and $|0\rangle$ respectively. The output field state ρ_1 can be found from (A.11), and is in general a mixed state.

For the purpose of eliciting the effect of propagation on TCS quantum noise reduction, it is simpler to calculate the fluctuation behavior of b directly from (3.4)-(3.7) and (3.14) than to use the transformed state (A.11). We find that

$$\langle b\rangle = \eta_1^{1/2}\tilde{\beta} , \qquad (3.15)$$

$$\langle b^\dagger b\rangle = \eta_1(|\tilde{\beta}|^2 + |\nu|^2) , \qquad (3.16)$$

$$\langle \Delta b_1^2\rangle = [\eta_1|\mu-\nu|^2 + (1-\eta_1)]/4 , \qquad (3.17)$$

$$\langle \Delta b_2^2\rangle = [\eta_1|\mu + \nu|^2 + (1-\eta_1)]/4 , \qquad (3.18)$$

where $b \equiv b_1 + jb_2$ defines the quadrature components of b.

Comparison of (3.6) and (3.18) reveals the following behavior: the TCS noise reduction that is manifest on a_1 when $|\mu-\nu|^2 < 1$ is nearly absent from b_1 unless $\eta_1 \simeq 1$. Physically, when $\eta_1 \ll 1$

the quantum characteristics of b are dominated by the vacuum coherent state properties of c through (3.14). Thus, to reap the communication benefits suggested by (3.6) we should arrange the propagation geometry so as to achieve a near unity fractional energy transfer (eigenvalue) for the TCS transmitter mode; such a situation will prevail in the classical near-field propagation regime [19]. Alternatively, for the classical far-field propagation regime in which $\eta_1 << 1$, we could conceivably improve on (3.17) by use of a TCS local oscillator near the receiver to cause the photon operator c in (3.14) to be in state $|\beta>_g$. These near field and far field cases are considered in Sections IV and V respectively.

IV Quantum Transmitter Optimization

The discussion surrounding (3.14)-(3.18) has the following generalization. Suppose a quantum transmitter produces a field operator $E_{A_0}(\bar{x},t)$ with state

$$\rho_{A_0} = |\rangle\langle| \bigotimes_n |0\rangle\langle 0| \,, \tag{4.1}$$

where $|\rangle$ is an arbitrary pure state. When $\eta_1 \ll 1$ the resulting state, ρ_1, for the first output eigenfunction in the expansion of $E_{A_L}(\bar{x},t)$ will be approximately a coherent state (insofar as field fluctuations are concerned)

$$\rho_1 \simeq |\eta_1^{1/2} \langle a\rangle\rangle\langle\eta_1^{1/2}\langle a\rangle| \tag{4.2}$$

in the absence of receiver state control, where a is the photon operator associated with the first transmitter mode. Evidently, with a field-measurement receiver, whatever benefits may be derived from optimum quantum state generation at the transmitter, they are limited to the regime wherein $\eta_1 \simeq 1$ when no receiver state control can be employed. It therefore behooves us to briefly address the eigenvalue behavior of the classical free-space mode decomposition before attempting a quantum transmitter theory.

IV.1 Near Field and Far Field Propagation

For the usual case in which the aperture diameters d_0 and d_L, and the path length L, are all much larger than the nominal wavelength λ_o, evanescence loss can be neglected and hence the Fresnel approximation

$$G_L(\bar{x},t;\bar{x}'t') = \delta(t-t'+L/c)$$
$$\cdot(-j\lambda_o L)^{-1} \exp[-j2\pi L(1+|\bar{x}-\bar{x}'|^2/2L^2)/\lambda_o] \ , \qquad (4.3)$$

can be used in place of (2.19). The mode decomposition (2.21) - (2.24) associated with the Fresnel approximation Green's function is well known [19], [20]; the salient features are summarized below.

Because of the delta function time dependence in (4.3), if we take $T_L = T_0 - L/c$ we obtain the factored mode decomposition $\{\zeta_n{}^0(\bar{x})\chi_m(t);\ \zeta_n{}^L(\bar{x})\chi_m(t-L/c);\eta_n\}$ where the temporal input modes $\{\chi_m(t)\}$ are an arbitrary CON set on T_0 and the spatial mode decomposition satisfies

$$\int_{A_0} d\bar{x}_2(d_L J_1(\pi d_L|\bar{x}_2-\bar{x}_1|/\lambda_o L)/2\lambda_o L|\bar{x}_2-\bar{x}_1|)\exp[j\pi(|\bar{x}_1|^2-|\bar{x}_2|^2)/\lambda_o L]$$
$$\cdot\zeta_n{}^0(\bar{x}_2) = \eta_n\zeta_n{}^0(\bar{x}_1) \ , \qquad (4.4)$$

$$\int_{A_0} d\bar{x}'\zeta_n{}^0(\bar{x}')(-j\lambda_o L)^{-1}\exp[-j2\pi L(1+|\bar{x}-\bar{x}'|^2/2L^2)/\lambda_o]$$

$$= \eta_n{}^{1/2}\zeta_n{}^L(\bar{x}) \ . \qquad (4.5)$$

Explicit results for $\{\zeta_n^{0}(\bar{x});\ \zeta_n^{L}(\bar{x});\ \eta_n\}$ are available [20]. The eigenvalues depend parametrically on the Fresnel number

$$D = (\pi d_0 d_L / 4\lambda_o L)^2 \tag{4.6}$$

and the input and output eigenfunctions are prolate spheroidal wavefunctions. For our purposes it is sufficient to note the asymptotic behavior of the eigenvalue distribution. Suppose the eigenvalues are arranged in decreasing order, then in the far-field region, $D \ll 1$, the maximum eigenvalue, η_1, is approximately equal to D, and all other eigenvalues are insignificant. When $D \gg 1$, near-field propagation prevails; there are then D eigenvalues near 1 and the rest are insignificant.

IV.2 Near-Field Transmitter Optimization

Consider the following problem of quantum state optimization in near-field free space communications. A continuous real-valued random variable α with density function $p(\alpha)$ is to be transmitted from $A_0 \times T_0$ by placing the photon annihilation operator a for the mode $\zeta_1^{0}(\bar{x})\chi_1(t)$ in state ρ_α. Let us assume that linear modulation is employed so that when α is chosen for transmission, ρ_α is constrained to satisfy

$$\langle a \rangle \equiv \mathrm{Tr}(\rho_\alpha a) = \alpha K , \tag{4.7}$$

where K is a positive constant. The receiver measures b_1, the

cosine quadrature of the photon annihilation operator b associated with the mode $\zeta_1^L(\bar{x})\chi_1(t-L/c)$ on $A_L \times T_L$. Because we are in the near-field propagation regime, we may assume $\eta_1 \simeq 1$, and hence $b \simeq a$, $b_1 \simeq a_1$. The problem is to choose ρ_α and K to maximize the average signal-to-noise ratio

$$SNR_{av} \equiv \frac{K^2\,\overline{\alpha^2}}{\int d\alpha p(\alpha) Tr[\rho_\alpha (\Delta a_1)^2]} \,, \tag{4.8}$$

where $\overline{\alpha^2}$ is the mean-square value of the parameter α, subject to the average photon number (energy) constraint

$$\int d\alpha p(\alpha) Tr(\rho_\alpha a^\dagger a) \leq N_s \,, \tag{4.9}$$

on the transmitter state. The classical average transmitted energy is

$$\int d\alpha p(\alpha) |Tr(\rho_\alpha a)|^2 = K^2 \overline{\alpha^2} \,. \tag{4.10}$$

Thus, because $\langle a^\dagger a\rangle \geq |\langle a\rangle|^2$, (4.9) is a more severe and more appropriate energy constraint for the quantum transmitter problem than the classical constraint $K^2\overline{\alpha^2} \leq N_s$.

We now show that the optimum state ρ_α^o and modulation constant K^o satisfy:

$$\rho_\alpha^o = |K^o\alpha(2N_s+1)^{1/2}\rangle_{gg}\langle K^o\alpha(2N_s+1)^{1/2}| \,, \tag{4.11}$$

$$K^o = [\overline{\alpha^2}(2N_s+1)/N_s(N_s+1)]^{-1/2} \,, \tag{4.12}$$

where the TCS $|K^o\alpha(2N_s+1)^{1/2}>_g$ has

$$\mu=(N_s+1)/(2N_s+1)^{1/2}, \quad \nu=N_s/(2N_s+1)^{1/2} . \tag{4.13}$$

That the optimization depends on $p(\alpha)$ only through $\overline{\alpha^2}$ should not be surprising because SNR_{av} is a second-moment quantity. The proof that (4.11)-(4.13) constitute optimum choices of ρ_α and K is as follows.

By means of (3.4)-(3.6) it is easy to show that (4.11)-(4.13) satisfy (4.7), (4.9) with equality, and give

$$SNR_{av}{}^o=4N_s(N_s+1) , \tag{4.14}$$

which we must show to be the optimum SNR_{av} value. We first re-write the energy constraint (4.9) in the equivalent form

$$Tr[\bar{\rho}(\Delta a_1)^2]+Tr[\bar{\rho}(\Delta a_2)^2]+K^2\overline{\alpha^2}\leq N_s+1/2 , \tag{4.15}$$

where

$$\bar{\rho}\equiv \int d\alpha p(\alpha)\rho_\alpha . \tag{4.16}$$

A simple convexity argument shows that $\rho_\alpha{}^o$ and K^o will satisfy (4.15) with equality. Moreover, when (4.15) is an equality (4.8) can be expressed as

$$SNR_{av}=(N_s+1/2-Tr[\bar{\rho}(\Delta a_2)^2])/Tr[\bar{\rho}(\Delta a_1)^2]-1 \tag{4.17}$$

Now, because $\bar{\rho}$ is a density operator we have the uncertainty relation

$$\mathrm{Tr}[\bar{\rho}(\Delta a_1)^2]\cdot\mathrm{Tr}[\bar{\rho}(\Delta a_2)^2]\geq 1/16 \ , \tag{4.18}$$

and, from (4.17), it is clear that $\rho_\alpha{}^o$ will satisfy (4.18) with equality. Under this minimum uncertainty condition we have

$$SNR_{av}=(N_s+1/2)/\mathrm{Tr}[\bar{\rho}(\Delta a_1)^2]-(4\mathrm{Tr}[\bar{\rho}(\Delta a_1)^2])^{-2}-1 \ , \tag{4.19}$$

which has the unconstrained maximum

$$SNR^o{}_{av}=4N_s(N_s+1) \ , \tag{4.20}$$

when

$$\mathrm{Tr}[\bar{\rho}(\Delta a_1)^2]=[4(2N_s+1)]^{-1} \ . \tag{4.21}$$

The proof is now complete in that (4.11)-(4.13) satisfy the constraints and achieve (4.19). Note that the use of ordinary coherent-state radiation in this linear modulation problem results in the maximized signal-to-noise ratio

$$SNR_{av}=4N_s \ , \tag{4.22}$$

when $K=(N_s/\overline{\alpha^2})^{1/2}$, $\rho_\alpha=|K\alpha\rangle\langle K\alpha|$.

For $N_s\gg 1$, the optimum TCS performance is vastly superior to the conventional-source result.

Note that we have not performed a combined transmitter-receiver optimization; the measurement of a_1 is fixed in advance. For linear modulation of TCS fields, however, one can easily show that the a_1 measurement actually minimizes the measurement variance for any input signal parameter. Thus, the performance (4.20) cannot be further improved by receiver optimization. The possibility is open that a joint transmitter-receiver optimization will lead to a superior performance than (4.20), which seems unlikely in the case of a continuous parameter α, but which is obviously true in the discrete case by transmitting photon eigenstates.

V Far-Field State Optimization

Recall from Section IV that under far-field propagation conditions the quantum field fluctuation characteristics of $E_{A_L}(\bar{x},t)$ will be dominated by the vacuum coherent-state behavior introduced by the radiative loss in the channel. In this section, we shall indicate how a TCS source might be used as a receiver local oscillator to achieve a state optimization analogous to that in Section IV.2. The present discussion is doubly speculative in that it assumes a source that is as yet unrealized (two-photon laser) and a spatial-mode mixing for which a practical configuration has yet to be found. However, the predicted performance increment is so large that continued implementation research is clearly justified. In particular, we shall find that by means of a TCS local oscillator we can increase post-measurement signal-to-noise ratio by a factor equal to the radiative loss of the channel.

Suppose, as in Section IV.2, that a continuous real-valued random variable α is to be transmitted using linear modulation from $A_0 \times T_0$ by placing the annihilation operator a for the mode $\zeta_1^0(\bar{x})\chi_1(t)$ in state ρ_α. The receiver measures b_1, the cosine quadrature of the annihilation operator b associated with the mode $\zeta_1^L(\bar{x})\chi_1(t-L/c)$ on $A_L \times T_L$. We shall assume far-field propagation prevails so that

$$b=\eta_1^{1/2}a+(1-\eta_1)^{1/2}c \tag{5.1}$$

where $\eta_1 \simeq (\pi d_0 d_L/4\lambda_0 L)^2 << 1$, and c is a photon annihilation operator whose state is unaffected by ρ_α. Note that any phase factor that would have multiplied $(1-\eta_1)^{1/2}c$ in (2.40) has been absorbed into the photon operator definition for c in (5.1). Our objective is to maximize the signal-to-noise ratio

$$SNR_{av} \equiv \frac{\int d\alpha\ p(\alpha)Tr[(\rho_1 b_1)^2]}{\int d\alpha\ p(\alpha)Tr[\rho_1(\Delta b_1)^2]} \qquad (5.2)$$

subject to the transmitter photon number constraint (4.9) by choice of ρ_α and K. In (5.2), ρ_1 is the state of b, which, in the absence of any radiation source other than the transmitter that produced E_{A_0} will be approximately the coherent state (c.f. (4.2))

$$\rho_1 = |\eta_1^{1/2}K\alpha\rangle\langle\eta_1^{1/2}K\alpha| \qquad (5.3)$$

Under (5.3) it is trivial to compute that the maximum signal-to-noise ratio

$$SNR_{av} = 4\eta_1 N_s \qquad (5.4)$$

results when we use the coherent-state $\rho_\alpha = |K\alpha\rangle\langle K\alpha|$ with $K = (N_s/\overline{\alpha^2})^{1/2}$. Comparison of (5.4) and (4.22) reveals the well known SNR_{av} attenuation that attends free-space propagation. To ameliorate this attenuation by means of state optimization, we must control the state of the photon operator c. The details are worked out below.

V.1 State Optimization by Spatial-Mode Mixing

For the assumed transmission geometry we can expand the z=L field operator in the series

$$E_L(\bar{x},t)=a\phi_s(\bar{x})\chi_1(t-L/c)+c\,\phi_R(\bar{x})\chi_1(t-L/c)+\sum_m a_m\xi_m(\bar{x},t) \quad , \quad (5.5)$$

where on Σ_L

$$\phi_s(\bar{x})=$$

$$\int_{A_0} d\bar{x}'\zeta_1^{\,0}(\bar{x}')(-j\lambda_o L)^{-1}\exp[-j2\pi L(1+|\bar{x}-\bar{x}'|^2/2L^2)/\lambda_o] \quad , \quad (5.6)$$

$$\phi_R(\bar{x})=\begin{cases} [(1-\eta_1)/\eta_1]^{1/2}\ \phi_s(\bar{x}) \ , & x\ \epsilon A_L \\ \\ -[\eta_1/(1-\eta_1)]^{1/2}\ \phi_s(\bar{x}) \ , & x\ \epsilon\Sigma_L-A_L \end{cases} \quad (5.7)$$

and $\{\phi_s(\bar{x})\chi_1(t-L/c),\ \phi_R(\bar{x})\chi_1(t-L/c),\ \xi_m(\bar{x},t)\}$ form a CON set on $\Sigma_L\times T_L$. Note that by virtue of (4.5), the restriction of the "signal" spatial mode, $\phi_s(\bar{x})$, to A_L is $\eta_1^{\,1/2}\ \zeta_1^{\,L}(\bar{x})$. By a simple calculation it follows from (5.7) and (2.8) that $\phi_R(\bar{x})$ is the other "received" spatial mode when we measure $\zeta_1^{\,L}(x)$ on A_L, i.e., a and c in (5.5) are the operators that appear in (5.1). Because these operators are associated with orthogonal modes on $\Sigma_L\times T_L$ we may take them to be quantum-mechanically independent, i.e., they are members of a set of photon operators obeying the CCR. Further, we suppose that when α is

to be transmitted the mode described by a is in a coherent state $\rho_\alpha=|(N_s/\overline{\alpha^2})^{1/2}\alpha\rangle\langle(N_s/\overline{\alpha^2})^{1/2}\alpha|$ (generated by the transmitter), and the mode described by c is in state ρ_c, controlled by a source (the local oscillator) in the vicinity of z=L. Our objective is to maximize the signal-to-noise ratio (5.2) subject to the local-oscillator energy constraint

$$Tr(\rho_c c^\dagger c)\leq N_R \ . \tag{5.8}$$

The preceding state optimization is straightfoward. We directly compute from (5.1), (5.2) that

$$SNR_{av}=\eta_1 N_s/[\eta_1/4+(1-\eta_1)Tr[\rho_c(\Delta c_1)^2]] \ . \tag{5.9}$$

Rewriting (5.8) as

$$Tr[\rho_c(\Delta c_1)^2]+Tr[\rho_c(\Delta c_2)^2]+|Tr(\rho_c c)|^2\leq N_R+1/2 \ , \tag{5.10}$$

the reasoning employed in Section IV.2 leads us to conclude that the TCS

$$\rho_c{}^o=|0\rangle_{gg}\langle 0| \ , \tag{5.11}$$

with

$$\mu=(N_R+1)^{1/2} \ , \ \nu=N_R{}^{1/2} \ , \tag{5.12}$$

yields the maximum signal-to-noise ratio

$$SNR_{av}{}^{o} = \frac{4N_s}{1+[(1+N_R)^{1/2}-N_R{}^{1/2}]^2(1-\eta_1)/\eta_1} . \tag{5.13}$$

When $N_R >> 1$, as will surely be the case, (5.13) becomes

$$SNR_{av}{}^{o} \simeq \frac{4N_s}{1+(1-\eta_1)/4\eta_1 N_R} . \tag{5.14}$$

Thus, when $N_R >> (1-\eta_1)/4\eta_1$ we find that the post-measurement signal-to-noise ratio for the far-field propagation channel with a TCS local-oscillator receiver reduces to the near-field propagation coherent-state transmitter result (recall (4.22))

$$SNR_{av}{}^{o} \simeq 4N_s . \tag{5.15}$$

Apparently use of the TCS local oscillator permits the post-measurement quantum noise to be attenuated to the point where the transmitter quantum fluctuations dominate the signal-to-noise ratio. One suspects, therefore, that additional benefits must accrue if we use a TCS transmitter as well as a TCS local oscillator, and indeed such is the case. We omit the general result for optimizing the signal-to-noise ratio (5.2) by choice of ρ_α, K, and ρ_c under the constraints (4.9) and (5.8) in favor of the following simpler special case. When

N_R satisfies $N_R \gg 1$, $N_R \gg (1-\eta_1)(N_s+1/2)/2\eta_1$ the optimum signal-to-noise ratio

$$SNR_{av}{}^{o} \simeq 4N_s(N_s+1) \tag{5.16}$$

is realized by using (4.11)-(4.13) for $\rho_\alpha{}^o$ and K^o and (5.11), (5.12) for $\rho_c{}^o$. Note that (5.16) equals the optimized near-field result (4.14).

V.2 Implementation of Spatial-Mode Mixing

Having found such a remarkable performance improvement is obtained through the beneficial spatial-mode mixing on A_L of a far-field propagation signal mode and a carefully chosen TCS receiver mode, we naturally inquire whether the results are specious. Let us say that there is some finite circular region, A_s, in the z=L plane which includes A_L and is such that

$$\int_{A_s} |\phi_s(\bar{x})|^2 d\bar{x} \simeq 1 \ , \tag{5.17}$$

i.e., A_s is the region occupied by the signal field in z=L. If our receiver measures the $\phi_s(\bar{x})$ mode on A_s rather than its restriction $\zeta_1{}^L(\bar{x})$ to A_L, we would not be surprised to obtain near-field signal-to-noise ratios, because (5.17) is precisely the condition that $\zeta_1{}^0(\bar{x})$ propagates from A_0 in z=0 to A_s in z=L without loss. The spatial-mode mixing analysis of Section V.1 therefore implies that the effect of the TCS local oscillator is to make the A_L measurement of b_1 equivalent to the A_s measurement of a_1. This is a hollow equivalence if to

generate the TCS mode $\phi_R(\bar{x})$ on A_s we require an apparatus that fills A_s. We need a technique to generate the desired field on A_s from a device that is significantly smaller than A_s.

As yet, we have not found such a technique, but neither have we found a general proof of its impossibility. The one result we have obtained is that the obvious answer, a divergent-beam local oscillator located in front of the receiver, cannot work. We present a brief treatment of the divergent beam system, because it illustrates the difficulties encountered in trying to realize the desired spatial-mode mixing on A_L. Whether or not any practical scheme exists for the TCS mode generation remains an open and important problem.

The geometry of the divergent-beam.receiver system is shown in Fig. 3. A TCS source located in the $z=L-\ell$ plane $(\ell<<L)$ radiates a diverging spatial mode $\phi_\ell(\bar{x})$ from a circular aperture A_ℓ that is much smaller than A_L. To achieve the desired mode mixing on A_L, $\phi_\ell(\bar{x})$ must give rise to the field $\phi_R(\bar{x})$ on A_s in the $z=L$ plane. According to the quantum propagation theorem, this propagation from A_ℓ to A_s must be (nearly) lossless or else the TCS quantum characteristics will not be preserved. Thus we require as a necessary condition the near-field propagation condition

$$(\pi d_\ell d_s/4\lambda_o \ell)^2 > 1 \qquad (5.18)$$

where d_ℓ and d_s are the diameters of A_ℓ and A_s. For typical parameter values, (5.18) does not pose a serious difficulty.

Unfortunately (5.18) is not a sufficient condition. It does not suffice, for our present purposes, that there is a mode $\phi_{\ell}(\bar{x})$ that suffers hardly any no radiative loss in propagation from A_{ℓ} to A_s. There must be a mode $\phi_{\ell}(\bar{x})$ on A_{ℓ} that yields without radiative loss a specific mode, determined from (5.7) with $\phi_s(\bar{x})$ modified to include the central obscuration of A_L by A_{ℓ}, on A_s (see Fig. 3). Simple recourse to Fresnel diffraction theory demonstrates that the above problem is over-constrained. The far-field propagation which governs the signal mode $\phi_s(\bar{x})$ gives rise to a quadratic phase curvature $\exp(-j\pi|\bar{x}|^2/\lambda_o L)$ in the z=L plane, and this same phase curvature persists in $\phi_R(\bar{x})$. The divergent TCS beam has a quadratic phase curvature $\exp(-j\pi|\bar{x}|^2/\lambda_o \ell)$ in the z=L plane. Because $\ell \ll L$ and $d_s^2/\lambda_o \ell \ll 1$ have already been assumed (the latter condition is implicit in (5.18)) the mode generated by $\phi_{\ell}(\bar{x})$ on the z=L plane will be almost orthogonal to $\phi_R(\bar{x})$.

Acknowledgment

The authors thank R.S. Kennedy for critical discussions and encouragement.

Appendix Transformation of Quantum States

Let $\underline{a}=\{a_j : 1 \leq j \leq N\}$ be a finite set of photon annihilation operators that obey the CCR. The density operator $\rho_{\underline{a}}$ describes the state of these modes, so that the state ρ_{a_i}, for the ith mode is obtained by tracing $\rho_{\underline{a}}$ over the other modes. The following joint anti-normally ordered characteristic function is always well defined [7], [21]

$$\chi_A^{\underline{a}}(\underline{\gamma},\underline{\gamma}^*)=\mathrm{Tr}[\rho_{\underline{a}} \exp(\sum_i \gamma_i{}^* a_i)\exp(-\sum_i \gamma_i a_i{}^\dagger)] , \tag{A.1}$$

where $\underline{\gamma}$ is a complex vector. Through normal-ordering techniques we can derive the following integral representation for $\rho_{\underline{a}}$ [7]

$$\rho_{\underline{a}}= \int \chi_A^{\underline{a}}(\underline{\gamma},\underline{\gamma}^*)\exp(-\sum_i \gamma_i a_i{}^\dagger)\exp(\sum_i \gamma^*{}_i a_i)d^2\underline{\gamma}/\pi^N . \tag{A.2}$$

Let $\underline{a}'=\{a_j' : 1 \leq j \leq N\}$ be another set of photon operators which are obtained from $\underline{a}$ by a unitary matrix transformation

$$\underline{a}'=\underset{\sim}{U}\underline{a} . \tag{A.3}$$

Because $\underline{a}$ obeys the CCR and $\underset{\sim}{U}$ is unitary, $\underline{a}'$ will also obey the CCR. The characteristic function, $\chi_A^{\underline{a}'}(\underline{\gamma},\underline{\gamma}^*)$ for $\underline{a}'$ can be computed from (A.1) and (A.3) with the result

$$\chi_A^{\underline{a}'}(\underline{\gamma},\underline{\gamma}^*)=\chi_A^{\underline{a}}(\underset{\sim}{U}^\dagger\underline{\gamma},(\underset{\sim}{U}^\dagger\underline{\gamma})^*) . . \tag{A.4}$$

Thus, the state $\rho_{\underline{a}'}$ of $\underline{a}'$ is determined by (A.2) and (A.4).

The above results can be formally extended to the case of an infinite number of modes. However, in actual applications there is usually some finite number of modes that suffices.

The quantum propagation theorem (2.33) requires the first half of the following unitary transformation

$$\begin{bmatrix} b_1 \\ \vdots \\ b_n \\ \vdots \\ \hline b_1' \\ \vdots \\ b_n' \\ \vdots \end{bmatrix} = \left[\begin{array}{cccc|cccc} \eta_1^{1/2} & & & 0 & (1-\eta_1)^{1/2} & & & 0 \\ & \ddots & & & & \ddots & & \\ 0 & & \eta_n^{1/2} & & 0 & & (1-\eta_n)^{1/2} & \\ & & & \ddots & & & & \ddots \\ \hline -(1-\eta_1)^{1/2} & & & 0 & \eta_1^{1/2} & & & 0 \\ & \ddots & & & & \ddots & & \\ 0 & & -(1-\eta_n)^{1/2} & & 0 & & \eta_n^{1/2} & \\ & & & \ddots & & & & \ddots \end{array}\right] \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ \vdots \\ \hline c_1 \\ \vdots \\ c_n \\ \vdots \end{bmatrix} \tag{A.5}$$

where b_n, b_n', a_n, c_n all obey the CCR. Under the condition

$$\rho_{\underline{a},\underline{c}} = \bigotimes_n \rho_{a_n} \bigotimes_n \rho_{c_n} , \tag{A.6}$$

where $\underline{a} \equiv \{a_j\}$, $\underline{c} \equiv \{c_j\}$, we have from (A.4)

$$\rho_{\underline{b}} = \bigotimes_n \rho_{b_n} , \tag{A.7}$$

where $\underline{b} \equiv \{b_j\}$, and for each n ρ_{b_n} can be computed from part of the two-dimensional transformation

$$\underline{b} = \begin{bmatrix} b \\ \\ b' \end{bmatrix} = \begin{bmatrix} \eta^{1/2} & (1-\eta)^{1/2} \\ \\ -(1-\eta)^{1/2} & \eta^{1/2} \end{bmatrix} \begin{bmatrix} a \\ \\ c \end{bmatrix} , \tag{A.8}$$

where $0 \leq \eta \leq 1$. In this case, (A.4) yields

$$\chi_A^b(\gamma,\gamma^*) = \chi_A^a(\eta^{1/2}\gamma,\eta^{1/2}\gamma^*)\chi_A^c((1-\eta)^{1/2}\gamma,(1-\eta)^{1/2}\gamma^*) . \tag{A.9}$$

In terms of the convenient density function $\rho^{(n)}(\alpha,\alpha^*) \equiv \langle\alpha|\rho|\alpha\rangle$ for coherent states $|\alpha\rangle$, we have from (A.9) and (A.2) the convolution formula

$$\rho_b^{(n)}(\alpha,\alpha^*)$$

$$\int \rho_a^{(n)}(\beta,\beta^*)\rho_c^{(n)}((1-\eta)^{-1/2}(\alpha-\eta^{1/2}\beta),(1-\eta)^{-1/2}(\alpha-\eta^{1/2}\beta)^*)d^2\beta/\pi(1-\eta) . \quad \text{(A.}$$

As an example of these transformation rules, consider the case in which a is in the TCS $|0\rangle_g$ with parameters μ_a, ν_a, and c is in the TCS $|0\rangle_g$ with parameters μ_c, ν_c. Then b, from (A.8), (A.9), has the characteristic function

$$\chi_A^b(\gamma,\gamma^*) = \exp(-\underline{\gamma}_q^T \Lambda \underline{\gamma}_q/2) \ , \qquad \text{(A.11)}$$

where $\underline{\gamma}=\gamma_1+j\gamma_2$, $\underline{\gamma}_q = \begin{bmatrix} 2\gamma_2 \\ -2\gamma_1 \end{bmatrix}$; (A.12)

$$\Lambda = \Omega^{-1} \left[\begin{array}{c|c} -\mu_a^* \nu_a \eta - \mu_c^* \nu_c (1-\eta) & |\mu_a|^2 \eta + |\mu_c|^2 (1-\eta) \\ \hline |\mu_a|^2 + |\mu_c|^2 (1-\eta) & -\mu_a \nu_a^* \eta - \mu_c \nu_c^* (1-\eta) \end{array}\right] (\Omega^{-1})^T \ , \qquad \text{(A.13)}$$

$$\Omega^{-1} = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} 1/2 \ . \qquad \text{(A.14)}$$

Equation (A.11) represents a TCS with added Gaussian noise and so is a mixed state in general. It can be shown that (A.11) becomes a pure state if and only if $\mu_a \nu_a \mu_c^* \nu_c^*$ is real.

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** Also Department of Electrical Engineering and Computer Science, M.I.T., Cambridge, Mass. 02139.

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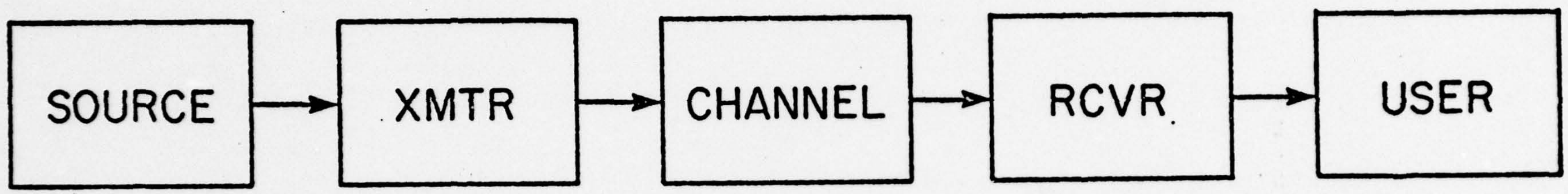
Figure Captions

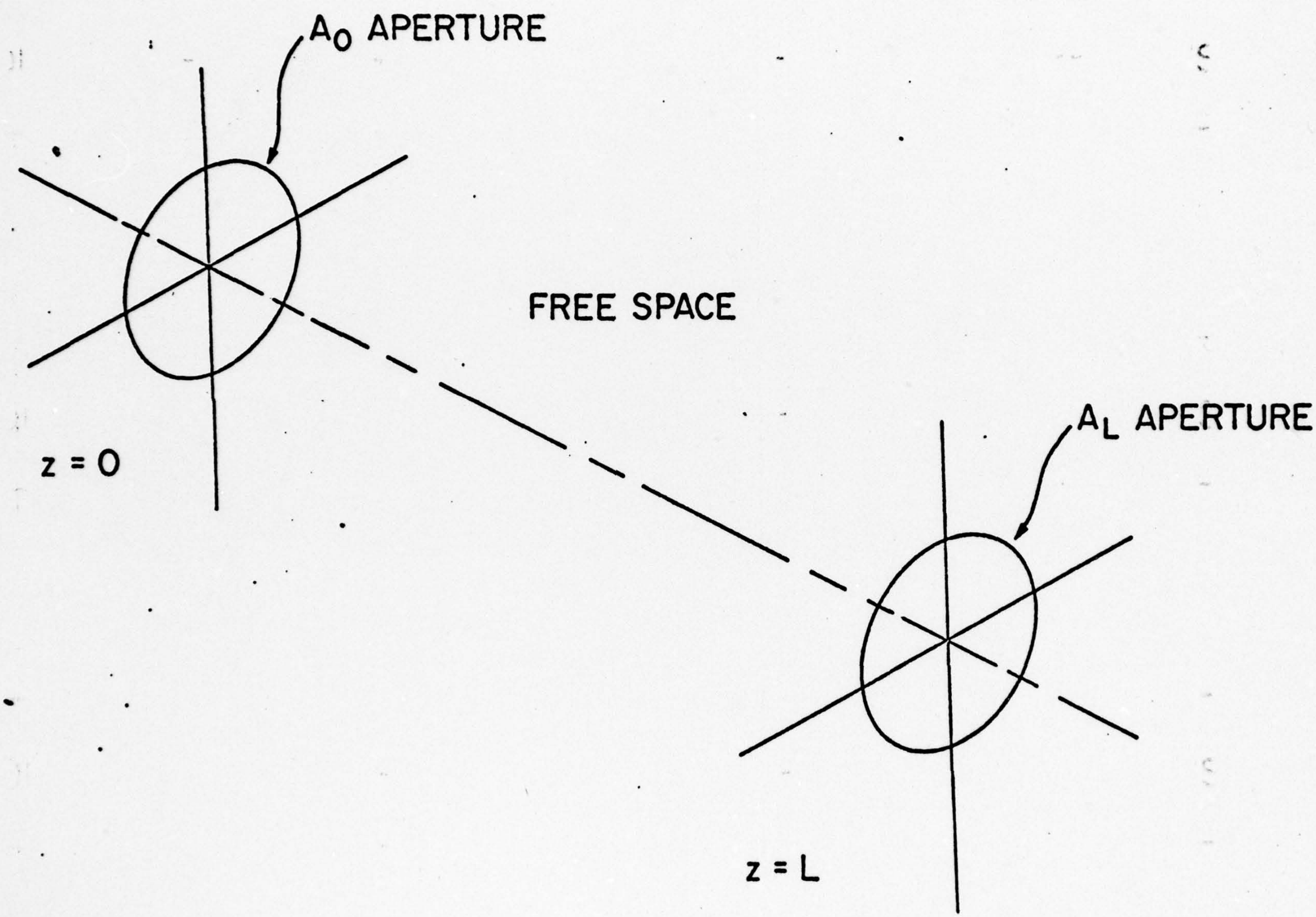
Fig. 1 Generic block diagram for a single communication link.

Fig. 2 Free space propagation geometry.

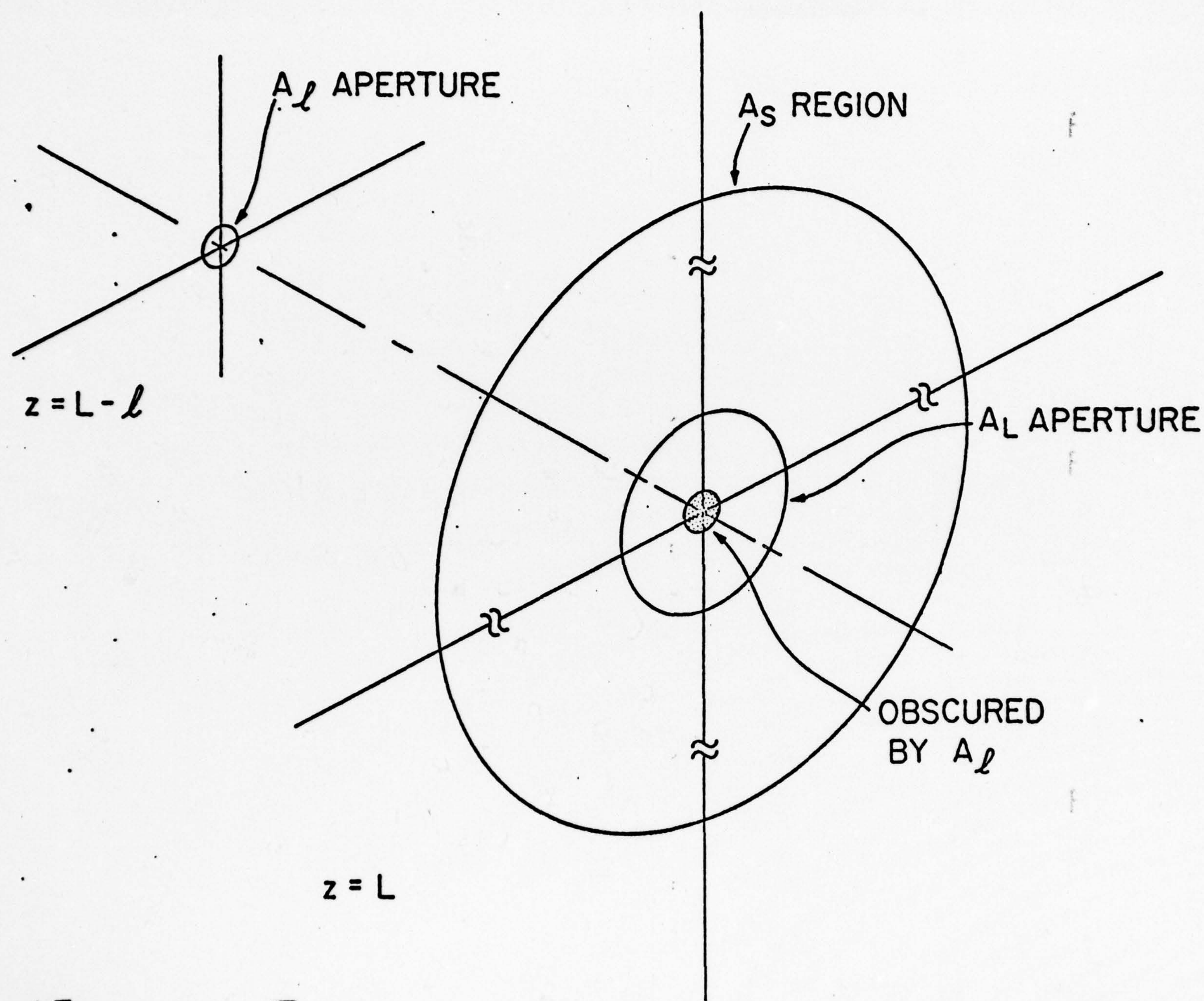
Fig. 3 Divergent-beam local oscillator geometry. TCS local oscillator radiates a diverging beam from the A_ℓ aperture in $z=L-\ell$, where A_ℓ is much smaller than A_L and ℓ is much smaller tnan L. This local-oscillator aperture prevents any signal light from reaching the central portion of A_L.



YUEN & SHAPIRO FIG. 1



YUEN & SHAPIRO FIG. 2



YUEN & SHAPIRO FIG. 3

in Abstracts of Papers, Part I, Fourth International Symposium on Information Theory, Leningrad, 1976.

TWO-PHOTON COHERENT STATES AND STATE OPTIMIZATION IN QUANTUM COMMUNICATION THEORY

H. P. YUEN
U.S.A.

The problems of quantum state optimization in optical communication are illustrated in some simplified cases. Under suitable conditions, greatly improved communication can be obtained via the use of two-photon coherent states which may be generated from ideal two-photon lasers.

In the quantum theory of communication, attention has so far been concentrated on the problems of optimum quantum measurements [1]-[3]. If one is interested in the ultimate performance limit of optical communication systems as imposed by quantum effects, it is essential to consider the problems of optimum quantum state generation [4], as the statistical description of the system is determined by both the quantum state and the quantum measurement. In the following we discuss a few problems of quantum state optimization which lead to a new class of radiation states different from the ordinary coherent states of Glauber [5]. These new states will be called "two-photon coherent states" (TCS) [previously referred to as "generalized coherent states" by the author [4], [6]]. They describe ideal two-photon laser radiation [4], [6], i.e., radiation obtained from stimulated two-photon emission. While no two-photon laser has actually been built, its realization may lead to significant performance improvement compared to ordinary coherent states as shown in the following.

Let $|\psi\rangle$ be an arbitrary quantum state of a radiation mode with photon annihilation operator a. We write $a = a_1 + ia_2$ for selfadjoint a_1, a_2. Assume that a_1 is measured at the receiver in state $|\psi\rangle$, as it may occur in the problem of transmitting continuous information. Let the mean $\langle\psi|a_1|\psi\rangle$ be a measure of the signal level, and define the signal-to-noise ratio

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$$\left(\frac{S}{N}\right)_{|\psi\rangle} \equiv \frac{\langle\psi|a_1|\psi\rangle^2}{\langle\psi|(\Delta a_1)^2|\psi\rangle} \tag{1}$$

with $a_1 = a_1 - \langle\psi|a_1|\psi\rangle$. Under the power constraint

$$\langle\psi|a^\dagger a|\psi\rangle < N_s \tag{2}$$

the optimum $(S/N)_{|\psi\rangle}$ achievable by any $|\psi\rangle$ is

$$(S/N)_0 = 4N_s(N_s + 1) \quad . \tag{3}$$

Equation (3) can be derived by elementary methods using the Heisenberg uncertainty principle.

Let $|\beta\rangle_g$ be the eigenstates of $b \equiv \mu a + \nu a^\dagger$ with eigenvalues β, $|\mu|^2 - |\nu|^2 = 1$. When $\nu = 0$, $|\beta\rangle_g$ becomes the ordinary coherent states. The detailed properties of these TCS $|\beta\rangle_g$ have been derived and the required information here can be found in [4] or [6]. With

$$\mu = (N_s+1)/\sqrt{2N_s+1} \quad , \quad \nu = N_s/\sqrt{2N_s+1} \quad , \quad \beta = \sqrt{N_s(N_s+1)} \tag{4}$$

one finds that $(S/N)_{|\beta\rangle_g}$ is given by the maximum value of (3). Thus, (1) is maximized by a TCS specified by (4).

Similarly, if an a priori distribution $p(s)$ is assigned to $s \equiv \langle\psi_s|a_1|\psi_s\rangle$ and the average signal-to-noise ratio

$$\left(\frac{S}{N}\right)_{av} \equiv \frac{\int s^2 p(s)\, ds}{\int ds\, p(s)\langle\psi_s|\Delta a_1{}^2|\psi_s\rangle} \tag{5}$$

is to be maximized by choosing $p(s)$, $|\psi_s\rangle$ under

$$\int ds\, p(s)\, \langle\psi_s|a^\dagger a|\psi_s\rangle < N_s \tag{6}$$

we find that the maximizing $|\psi_s\rangle$ is

$$|\psi_s\rangle_0 = |s\sqrt{2N_s+1}\rangle_g \quad . \tag{7}$$

Any density function $p_0(s)$ with variance

$$\sigma_s{}^2 = N_s(N_s+1)/(2N_s+1) \tag{8}$$

is optimum, with resulting maximum $(S/N)_{av}$ given again by (3). If $p(s)$ is fixed with given variance $\sigma_s{}^2 < N_s$, (5) is also maximized under (6) by TCS (different from (7) in general).

Note from (3) that $(S/N)_{|\beta\rangle_g}$ can be much higher than the value $4N_s$ obtained from coherent states under the same constraint (2) or (6). However, the above optimization does not yet imply that $|\beta\rangle_g$ minimizes the mean-square error in an analog estimation problem. It only implies that $|\beta\rangle_g$ minimizes the error among those states which give a Gaussian density function when a_1 is

[illegible] ... results on mean-square error measurement optimization for [illegible] states $|\beta\rangle_g$ have been given by Holevo [7].

The application of (7)-(8) with the Gaussian $p_0(s) = (1/\sqrt{2\pi}\sigma_s)$ exp $(-s^2/2\sigma_s^2)$ and a_1 measurement lead to the following information capacity [4]

$$C = \log (1+2N_s) \quad . \tag{9}$$

Equation (9) may be compared with the heterodyne and homodyne capacities for coherent states [8],

$$C > C_{he} = \log (1+N_s) \quad , \quad C > C_{ho} = \frac{1}{2} \log (1+4N_s) \quad . \tag{10}$$

By a conventional nonlinear programming optimization, it may be shown that (9) in fact maximizes the capacities among all the TCS and all the measurements described by $|\beta\rangle_g {}_g\langle\beta|$.

Our above discussion ignored the effect of the transmission medium on the quantum state. It is wellknown that the information capacity is maximized by photon eigenstates and photon counting [8], [9] in the absence of loss and other channel interferences. Under this *lossless free-space* situation, (9) is effectively equal to the optimum capacity for $N_s \gtrsim 10^2$. In actual communication, radiative loss is very significant and has to be taken into account. In this case the state at the receiver is determined largely by the receiver environment, and can be controlled at least to some extent. Under ideal conditions it is possible to recover the value (9) where N_s is the signal energy at the transmitter [4]! On the other hand, serious partition noise appears at such a receiver even if one can realistically generate photon eigenstates at the transmitter. The theoretical and practical possibilities of a *quantum-state-generating receiver* [4] *for controlling the receiver state* should be persued for improved optical communication.

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SIGNAL PROCESSING FOR TWO-PARAMETER RANDOM FIELDS

Eugene Wong

USA

Abstract

In this paper we present a summary of some recent results on the theory of martingales and stochastic integrals for process with a two-dimensional parameter, and examine some possible applications to problems in signal processing. Specifically, we shall consider the detection problem of testing between the two hypotheses

H_0: The observed process $X(t_1,t_2)$ is a Wiener process.

H: The observed process is of the form

$$X(t_1,t_2) = \int_0^{t_1}\int_0^{t_2} S(\tau_1,\tau_2)\, d\tau_1 d\tau_2 + W(t_1,t_2)$$

and W is a Wiener process.

A series of likelihood ratio formulas which fully generalize the exponential formula of one-parameter processes will be presented.

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